Lecture 5
Biosignal Analysis – 1

Digital Signal Processing and Analysis
in Biomedical Systems

Contents

- What and why we analyze signals?
- Unified approach and stages.
- Feature Extraction – 1
- Fourier Analysis

What are biosignals?

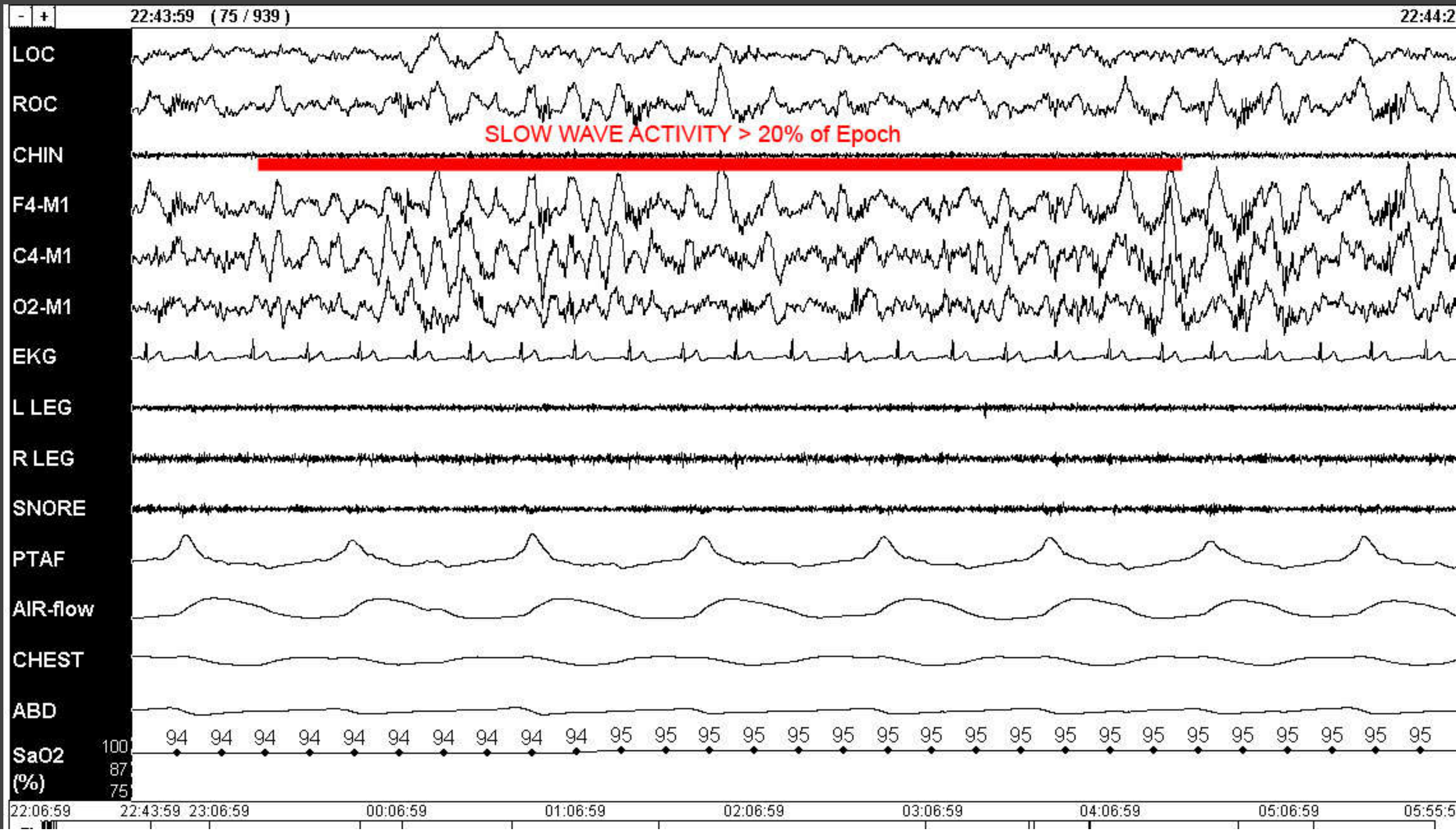
All types of biomedical systems either generate the signals to influence the human body, or analyze biosignals to extract useful information about functioning of human body.

Signal – is the parameter that is observable from the object.

Biosignal is a description of physiological phenomenon of any nature.

Bio+Signal = “living object” + “function that carries information about the behavior or state”. Biosignals are the key objects in Biosystems.

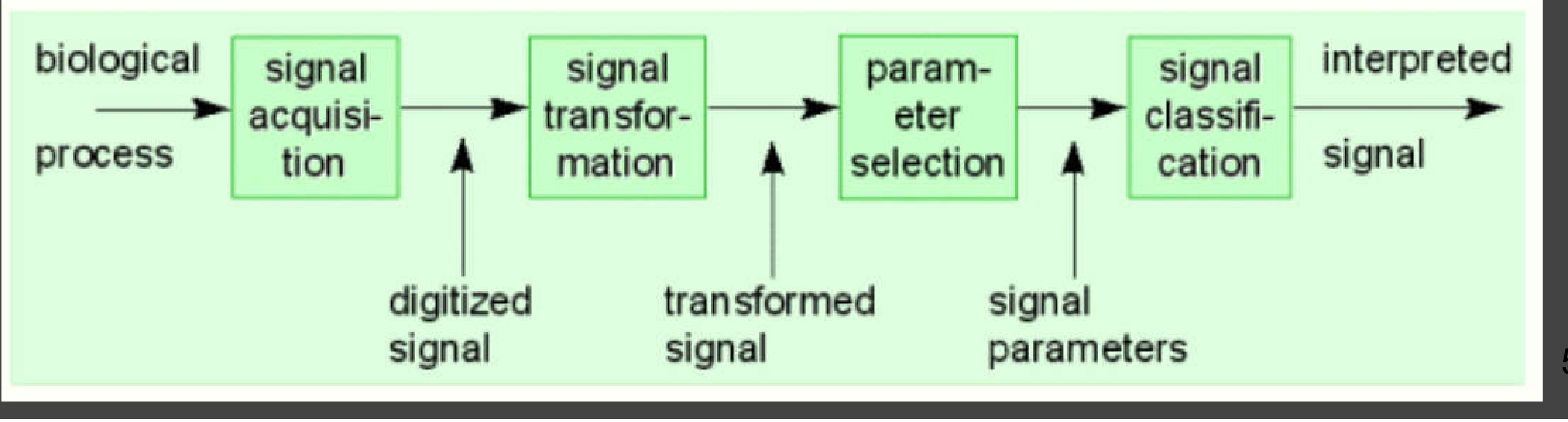
PSG signal


22:43:59 (75 / 939)
22:44:2
LOC
ROC
CHIN
SLOW WAVE ACTIVITY > 20% of Epoch
F4-M1
C4-M1
O2-M1
EKG
L LEG
R LEG
SNORE
PTAF
AIR-flow
CHEST
ABD
SaO2 (%)
100
87
75
94 94 94 94 94 94 94 94 94 94 95 95 95 95 95 95 95 95 95 95 95 95 95 95 95 95 95 95 95
22:06:59 22:43:59 23:06:59 00:06:59 01:06:59 02:06:59 03:06:59 04:06:59 05:06:59 05:55:5

Signal analysis

The purpose of signal analysis is to extract parameters of input signals to further convert it into information about the object:

- Diagnosis,
- Prognosis.



Analysis = Feature Extraction

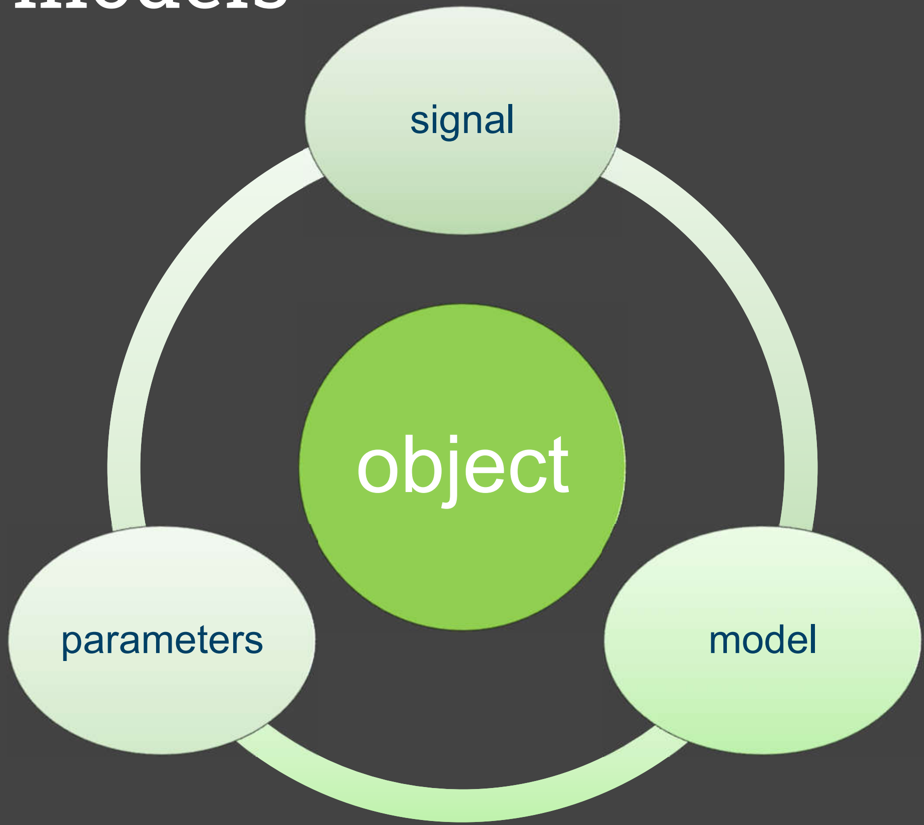
Signal **feature** is the measurable property of the signal being analysed.

Feature engineering – extraction, selection and construction of feature sets. It requires deep domain expertise and:

- Knowledge,
- Experimentation,
- Intuition.

Feature Learning – the techniques that transform raw data into the representation that can befurther used in ML tasks.

Signals and models



Features of (bio)signals

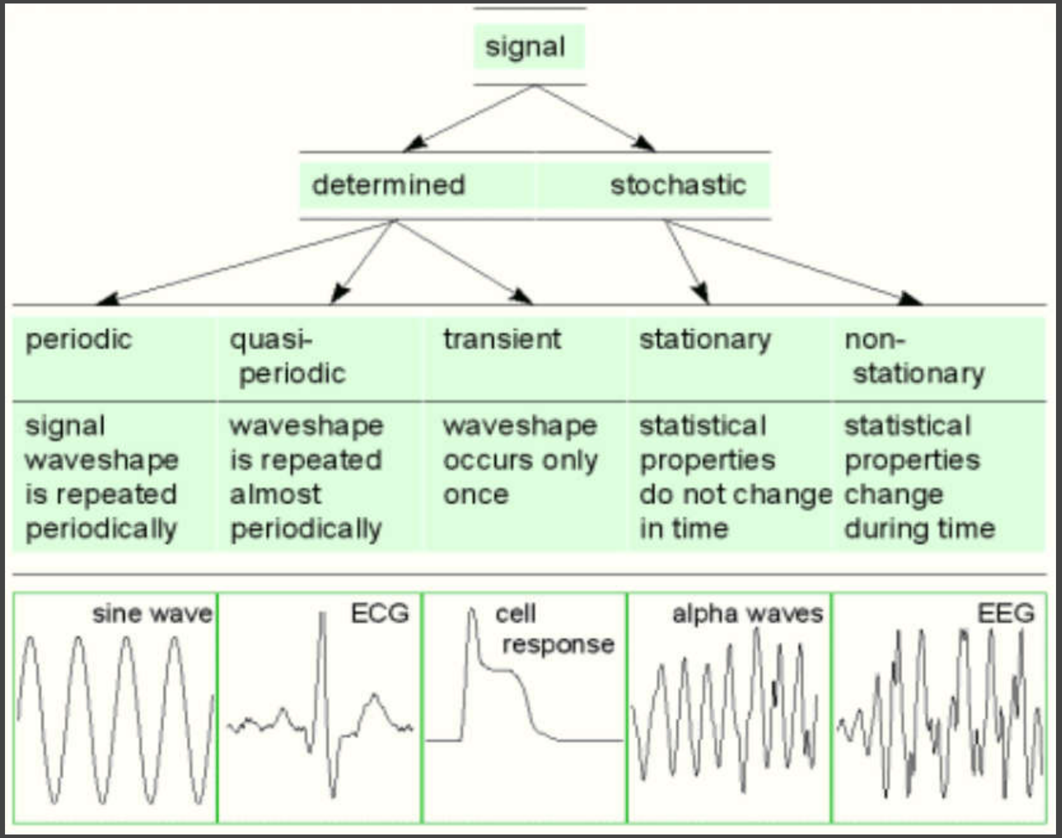
Feature extraction is based on the analysis of signal.

Selection of method for analysis:

- Type of process/object under analysis
- Type of signal
- Type of function

Information we need to extract from the signal = information about the process or object if interest

Types of biosignals



Signal Analysis approaches

- Linear / non-linear
- Univariate / multivariate

	Linear	Non-linear
Univariate	$R = F[s(t)]$, $F[as_1(t) + bs_2(t)] = aF[s_1(t)] + bF[s_2(t)]$	$R = F[s(t)]$, $F[as_1(t) + bs_2(t)] \neq aF[s_1(t)] + bF[s_2(t)]$
Multivariate	$R = F[s_1(t), s_2(t), s_3(t), \ldots, s_N(t)]$, $F[as_1(t) + bs_2(t)] = aF[s_1(t)] + bF[s_2(t)]$	$R = F[s_1(t), s_2(t), s_3(t), \ldots, s_N(t)]$, $F[as_1(t) + bs_2(t)] \neq aF[s_1(t)] + bF[s_2(t)]$

Linear Univariate analysis

- Fourier analysis
- Short-time Fourier transform
- Wavelet analysis

Fourier Analysis

Fourier decomposition is the representation of signals as a weighted sum of sinusoidal components, each with its own magnitude, frequency and initial phase.

Types of Fourier decompositions (continuous/discrete):

- Fourier series (for periodic infinite signals)
- Fourier Integral transform (for non-periodic, finite signals)

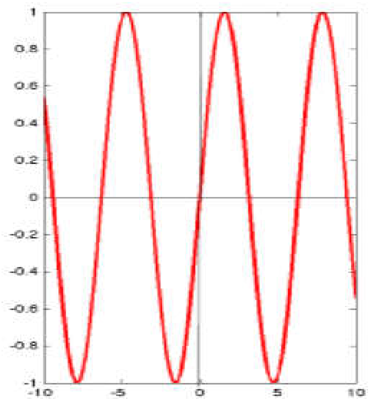
Fourier Series – 1

A function f is periodic, with period T if and only if:

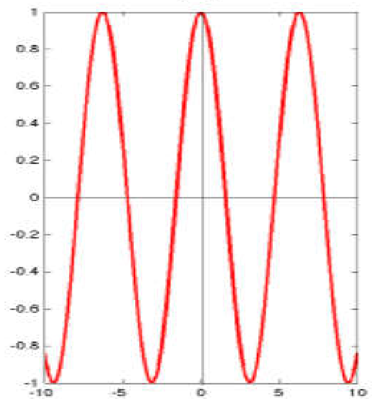
$$\forall x, \quad f(x+T) = f(x)$$

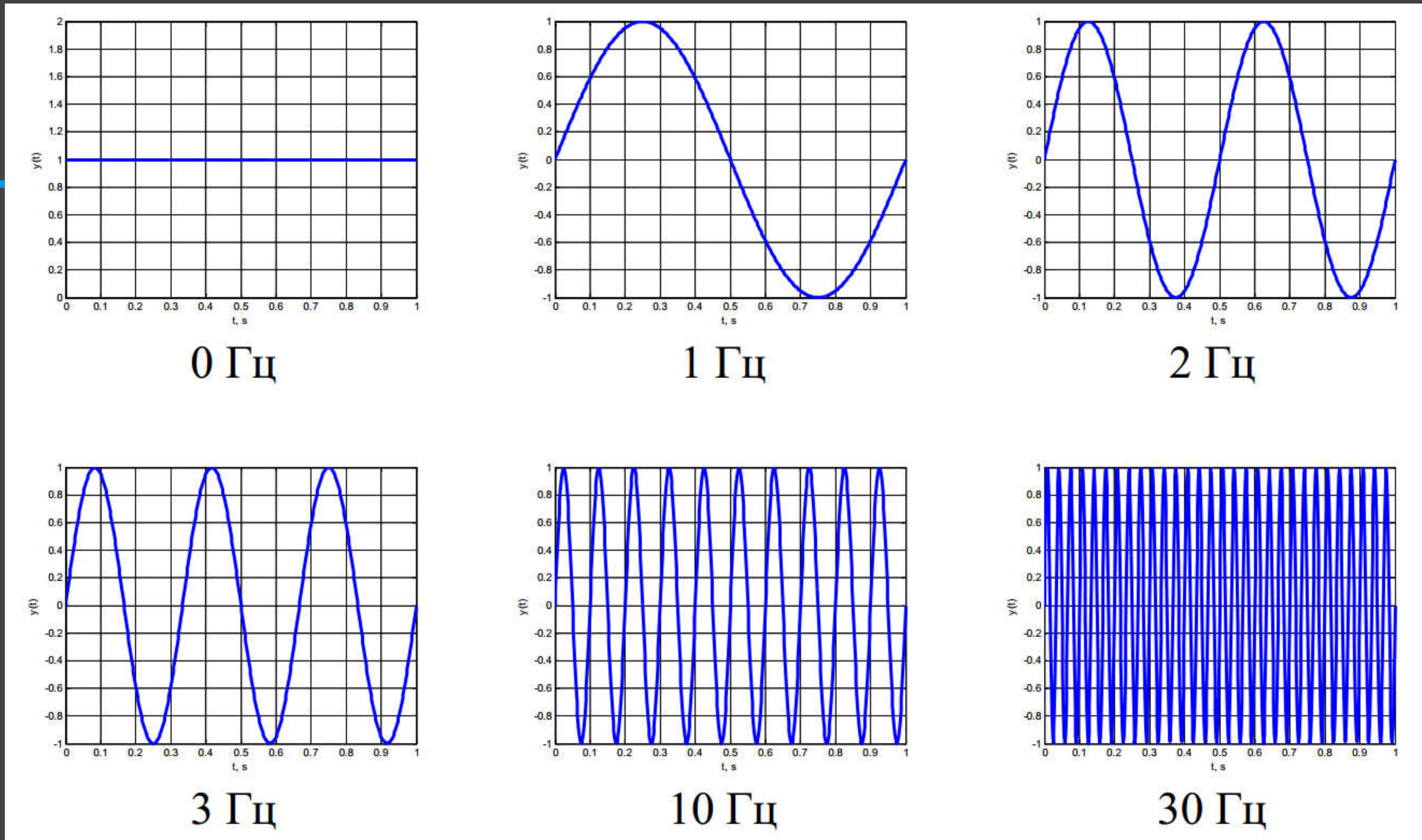
Examples of periodic functions:

sin(t)



cos(t)





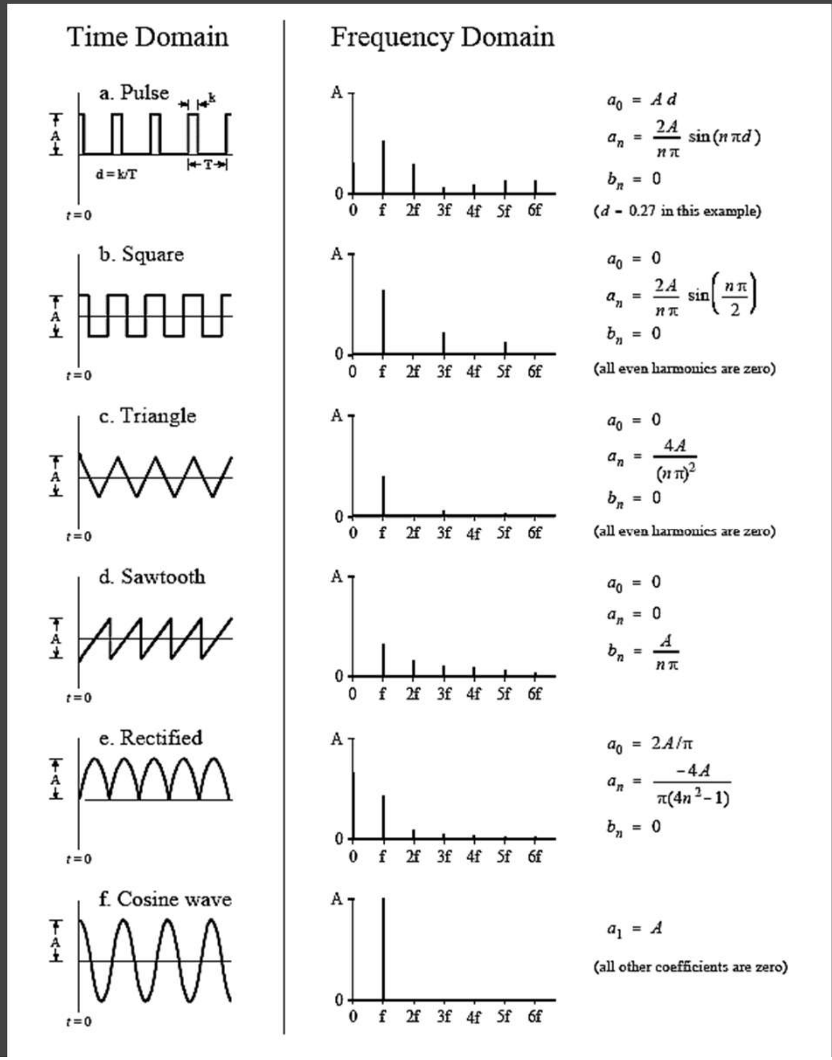
y(t)
t, s
0 Гц
1 Гц
2 Гц
3 Гц
10 Гц
30 Гц

Fourier Series – 2

A Fourier series of a periodic function f (with period 2π) defined as an expansion of f in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

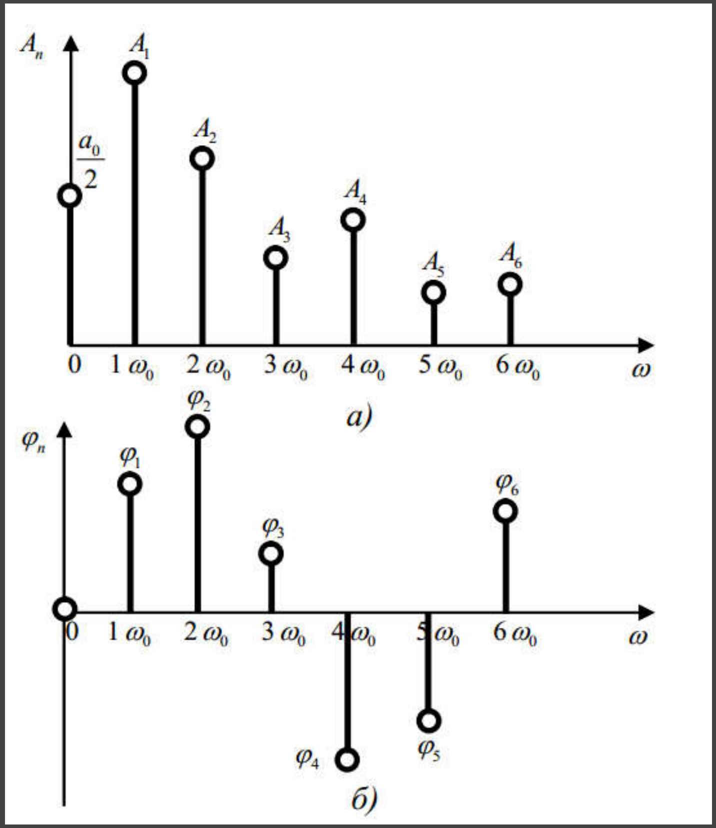
Fourier series are named in honor of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series.


Time Domain
Frequency Domain
a. Pulse
k
T
d = k/T
t = 0
A
0
0 f 2f 3f 4f 5f 6f
$a_0 = A d$
$a_n = \frac{2A}{n\pi} \sin(n\pi d)$
$b_n = 0$
(d = 0.27 in this example)
b. Square
$a_0 = 0$
$a_n = \frac{2A}{n\pi} \sin\left(\frac{n\pi}{2}\right)$
$b_n = 0$
(all even harmonics are zero)
c. Triangle
$a_0 = 0$
$a_n = \frac{4A}{(n\pi)^2}$
$b_n = 0$
(all even harmonics are zero)
d. Sawtooth
$a_0 = 0$
$a_n = 0$
$b_n = \frac{A}{n\pi}$
e. Rectified
$a_0 = 2A/\pi$
$a_n = \frac{-4A}{\pi(4n^2-1)}$
$b_n = 0$
f. Cosine wave
$a_1 = A$
(all other coefficients are zero)

Fourier Series – 3

$$x(t) = c_0 + \sum_{n=1}^{+\infty} 2|c_n|\cos(n\omega_0 t + \varphi_n)$$

Amplitude and Phase spectra



Integral Fourier Transform – 1

$$F(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

Integral Fourier Transform – 2

$$F(\omega) = R(\omega) + iI(\omega) = A(\omega)e^{i\Phi(\omega)}$$

$$A(\omega) = |F(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$$

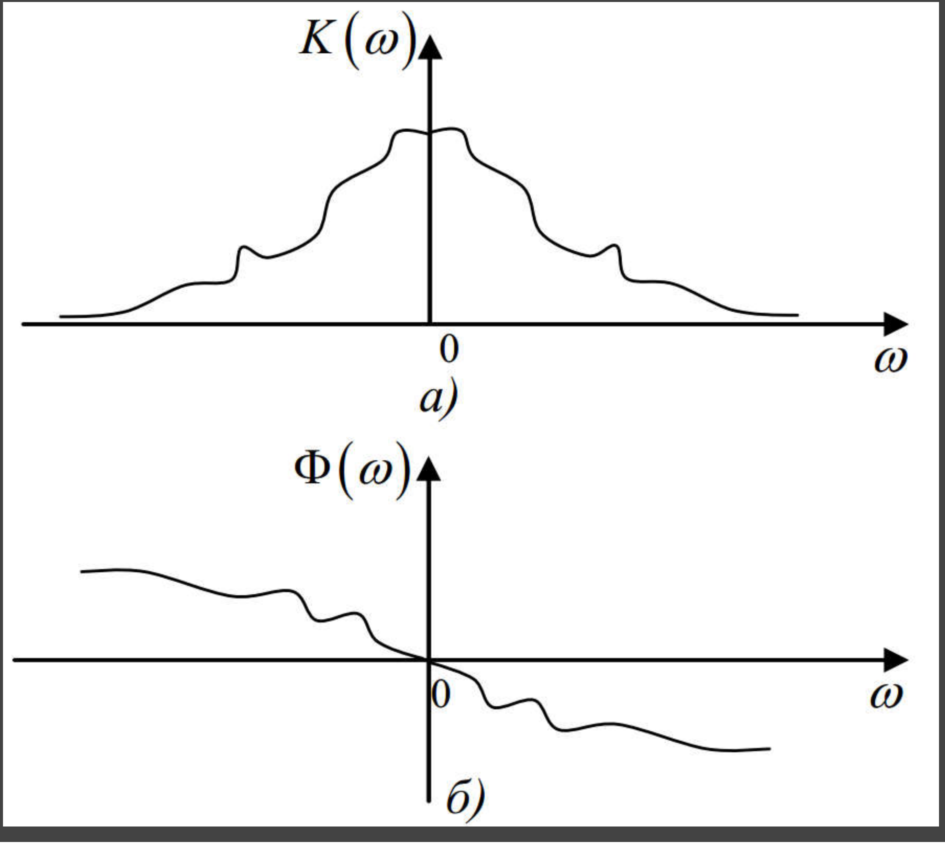
$$\Phi(\omega) = \arg F(\omega) = \arctan \frac{I(\omega)}{R(\omega)}$$

$A(\omega)$ **Amplitude spectrum**

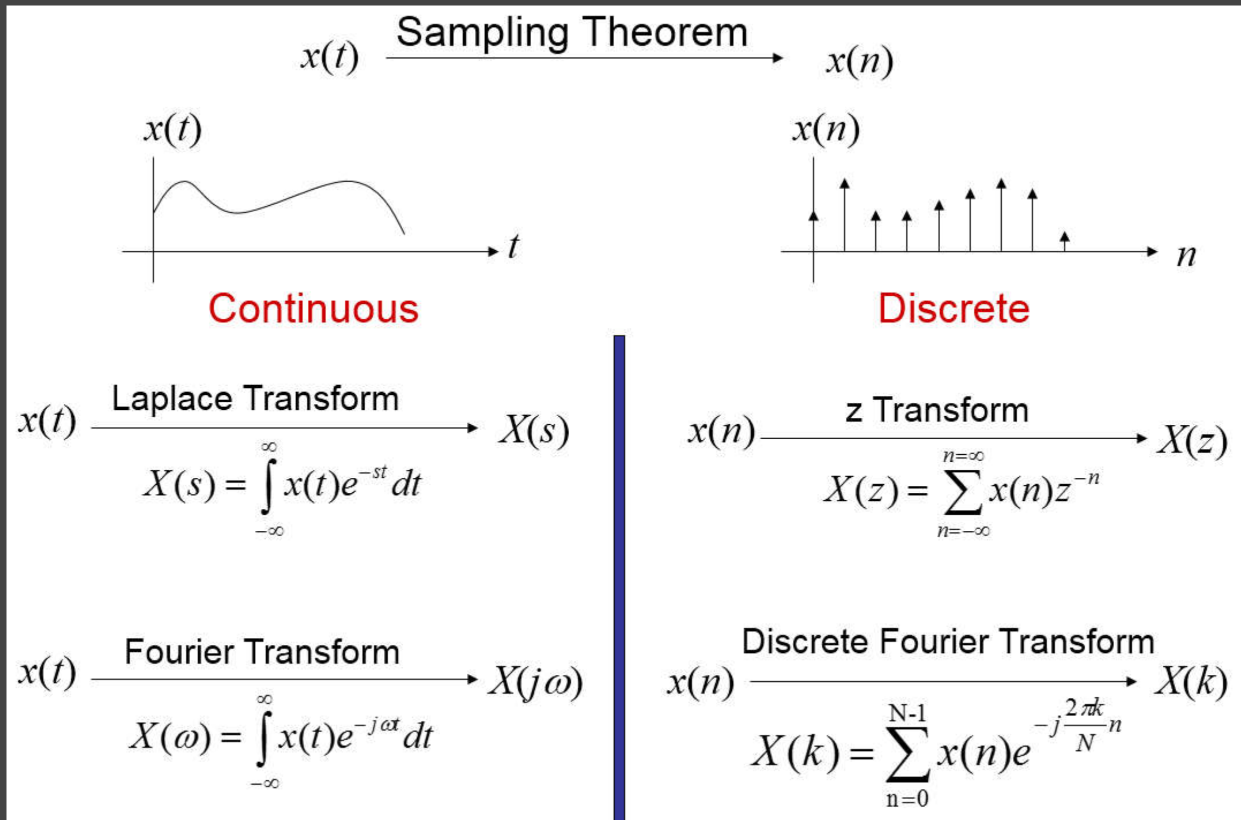
$\Phi(\omega)$ Phase spectrum

In most application it is the amplitude (or the power) spectrum that is of interest.

Integral Fourier Transform – 3



Digital Fourier Transform – 2



Digital Fourier Transform – 3

$$F_0[n] = \frac{1}{\sqrt{N}}$$

$$F_1[n] = \frac{1}{\sqrt{N}} e^{2\pi j \frac{n}{N}}$$

$$F_2[n] = \frac{1}{\sqrt{N}} e^{4\pi j \frac{n}{N}}$$

$$F_3[n] = \frac{1}{\sqrt{N}} e^{6\pi j \frac{n}{N}}$$

$$\ldots$$

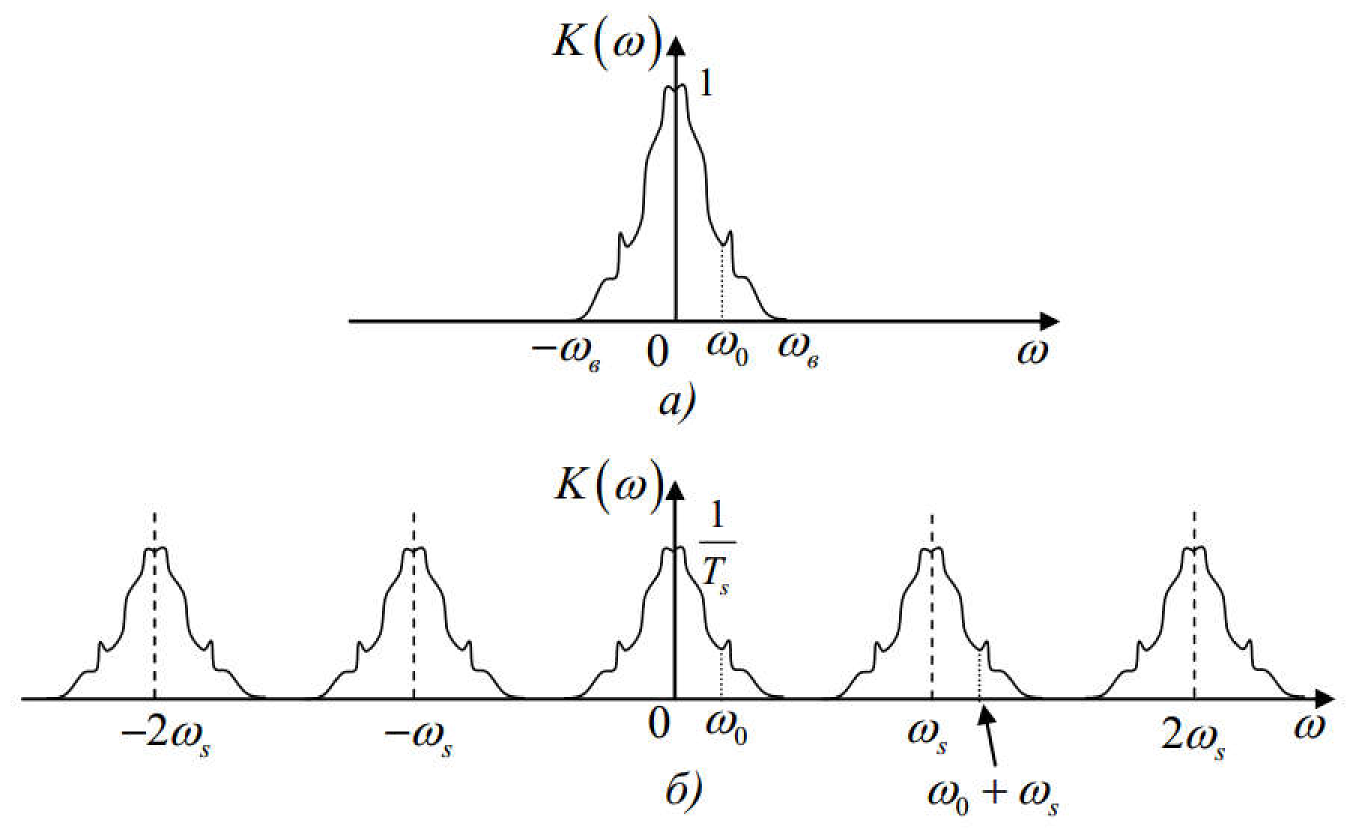
$$F_{N-1}[n] = \frac{1}{\sqrt{N}} F e^{2\pi j (N-1) \frac{n}{N}}$$

$$c[m] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-2\pi j \frac{m}{N} n}$$

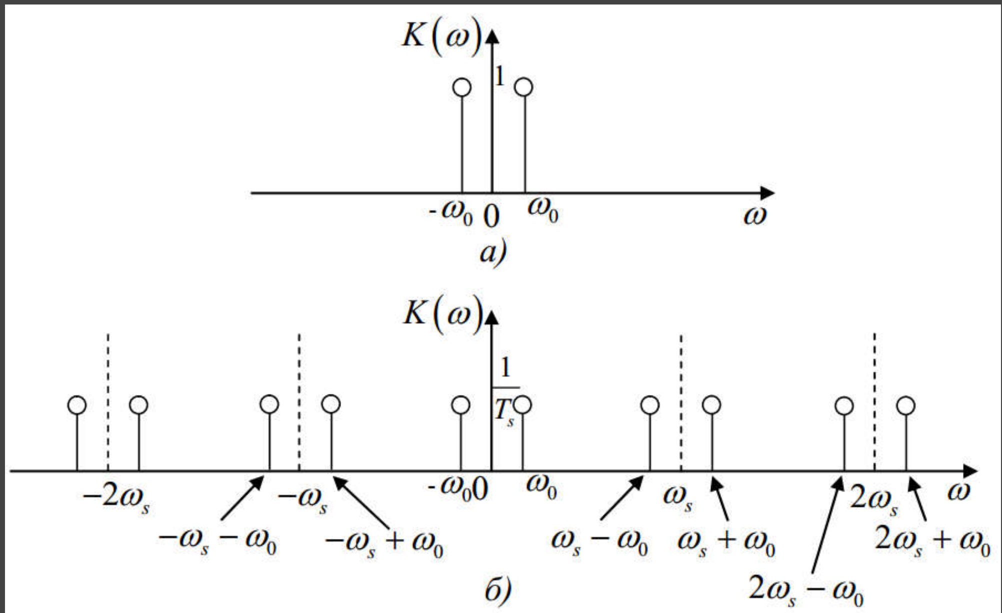
$$x[n] = \sum_{m=0}^{N-1} c[m] e^{2\pi j \frac{m}{N} n}$$

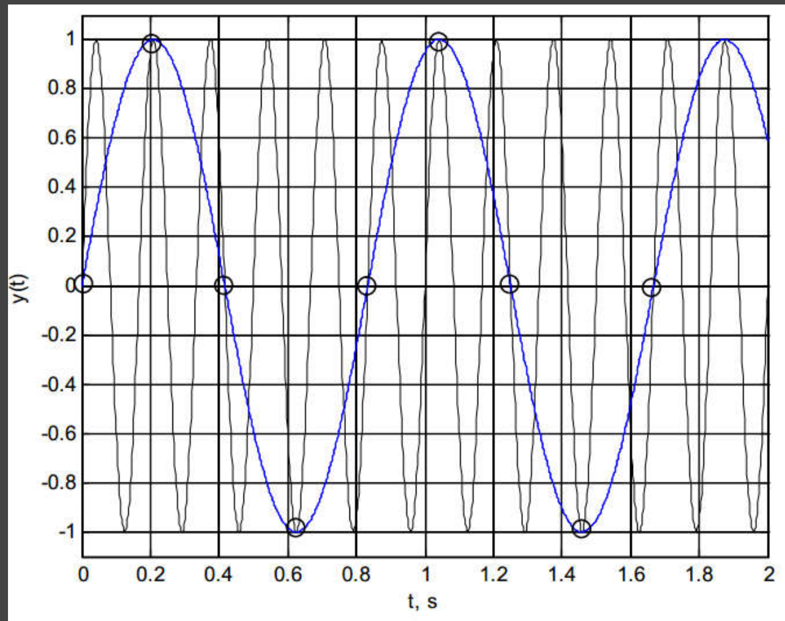
$$c[m] = |c[m]| e^{j \arg(c[m])} = K[m] e^{j\Phi[m]}$$

Digital Fourier Transform – 4

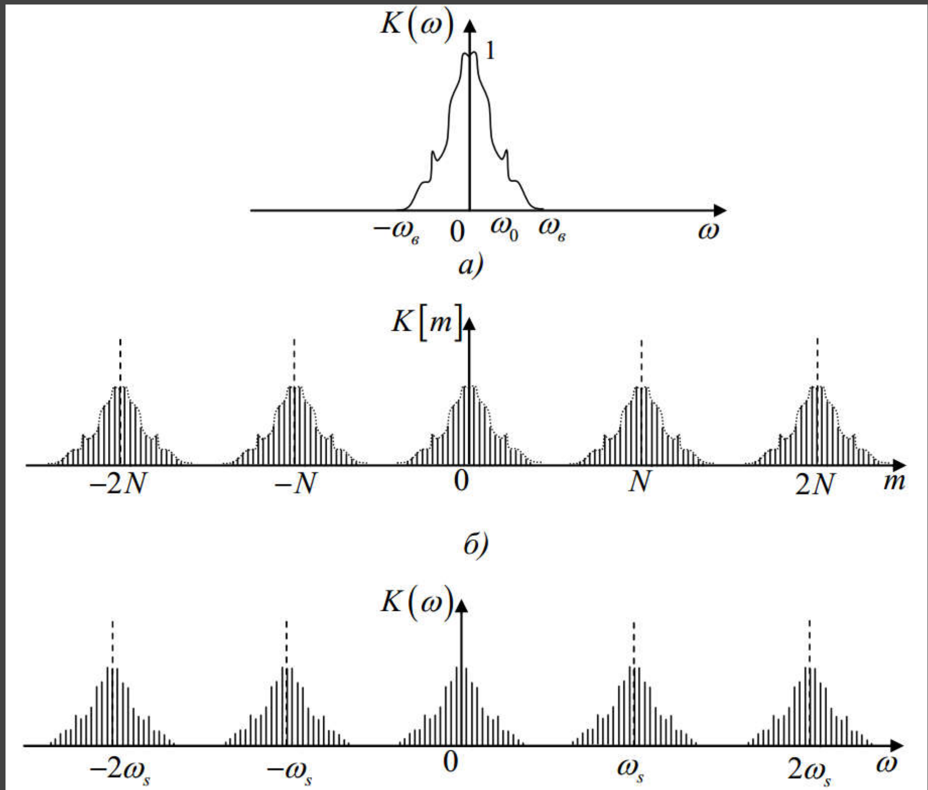


Digitization of sinusoidal wave

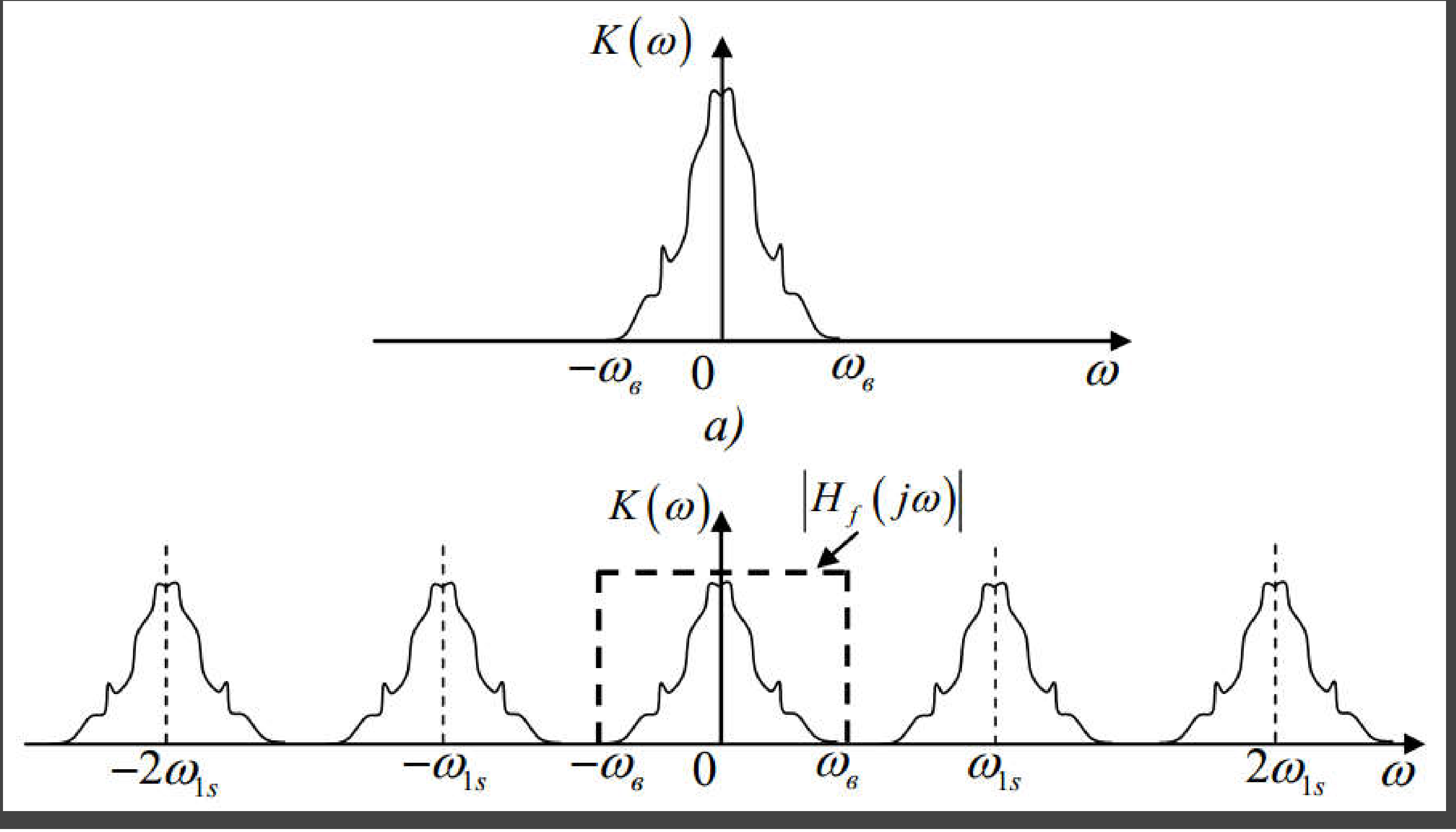




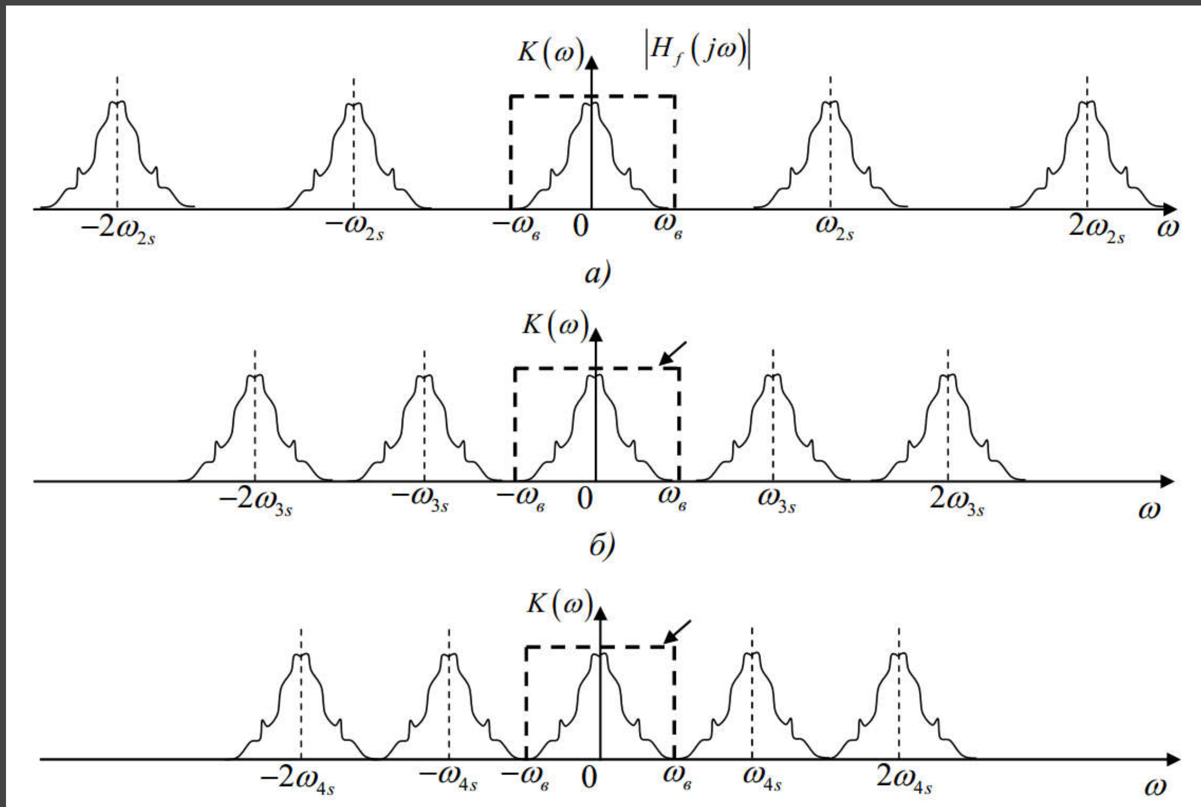
Digital Fourier Transform – 5



Aliasing – 1

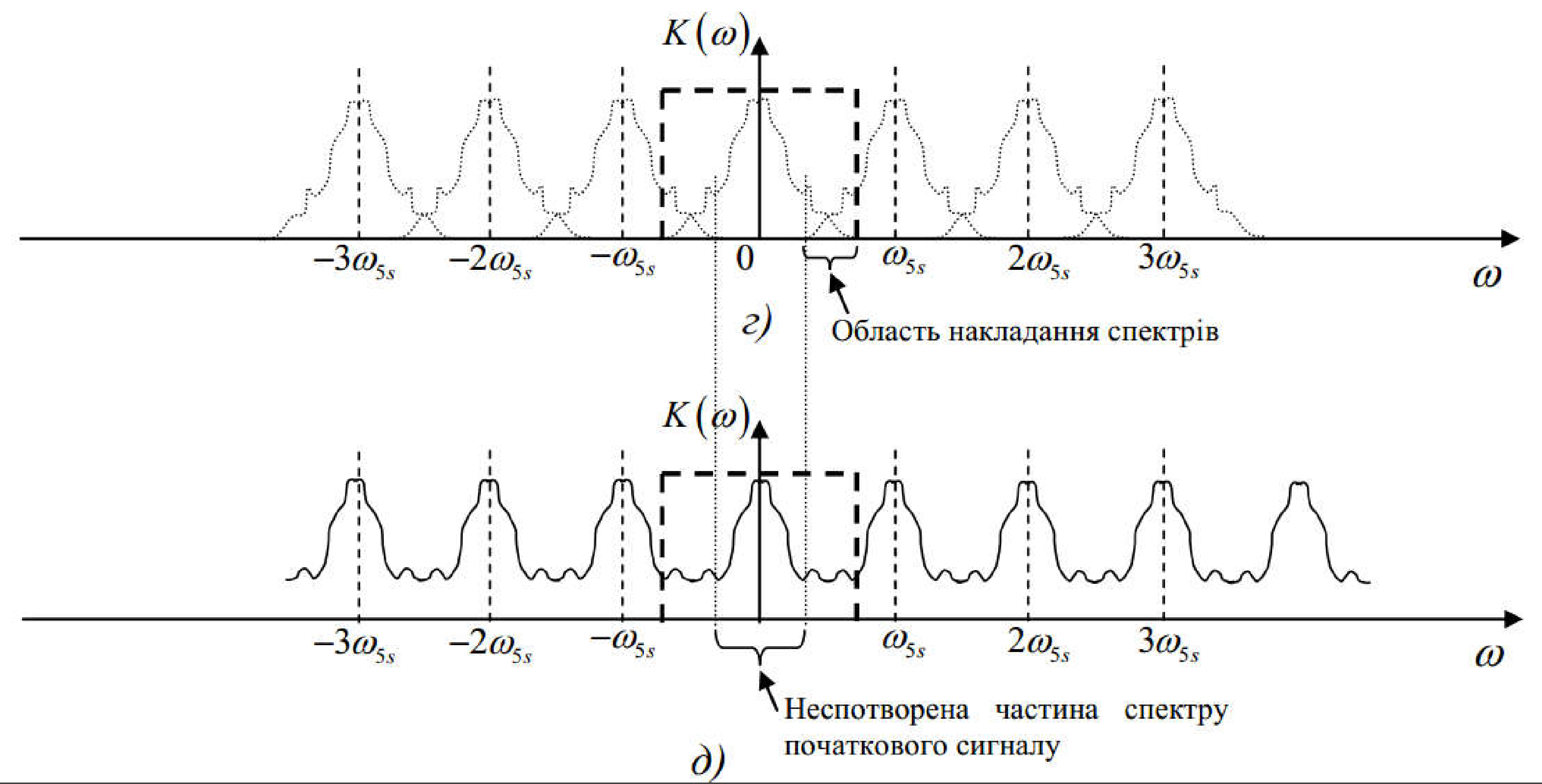


Aliasing – 2



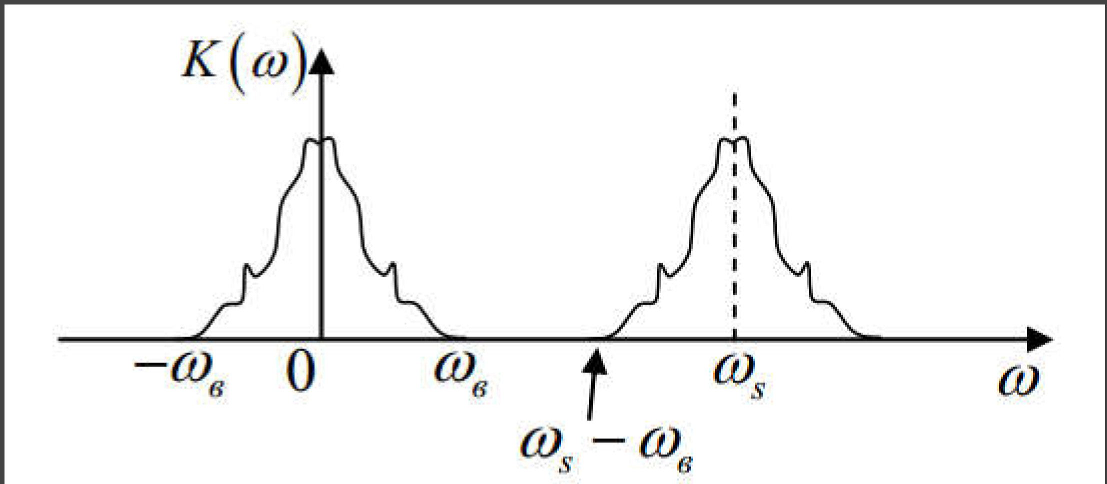
K(ω)
|H_f(jω)|
−2ω_2s
−ω_2s
−ω_в
0
ω_в
ω_2s
2ω_2s
ω
а)
K(ω)
−2ω_3s
−ω_3s
−ω_в
0
ω_в
ω_3s
2ω_3s
ω
б)
K(ω)
−2ω_4s
−ω_4s
−ω_в
0
ω_в
ω_4s
2ω_4s
ω

Aliasing – 3



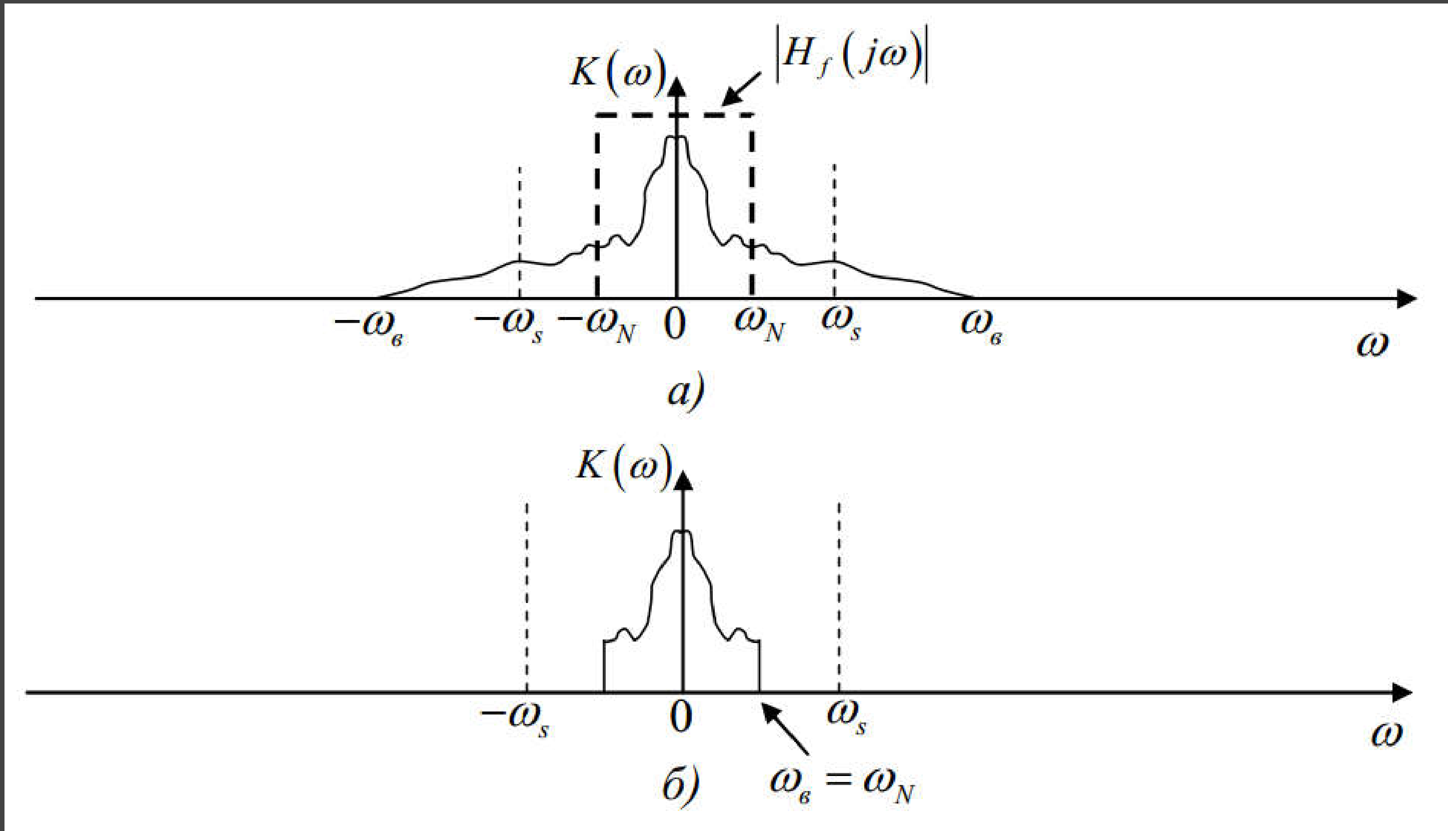
$K(\omega)$
$-3\omega_{5s}$
$-2\omega_{5s}$
$-\omega_{5s}$
0
ω_{5s}
$2\omega_{5s}$
$3\omega_{5s}$
ω
г)
Область накладання спектрів
$K(\omega)$
$-3\omega_{5s}$
$-2\omega_{5s}$
$-\omega_{5s}$
ω_{5s}
$2\omega_{5s}$
$3\omega_{5s}$
ω
Неспотворена частина спектру початкового сигналу
д)

Kotelnikov-Shannon Theorem

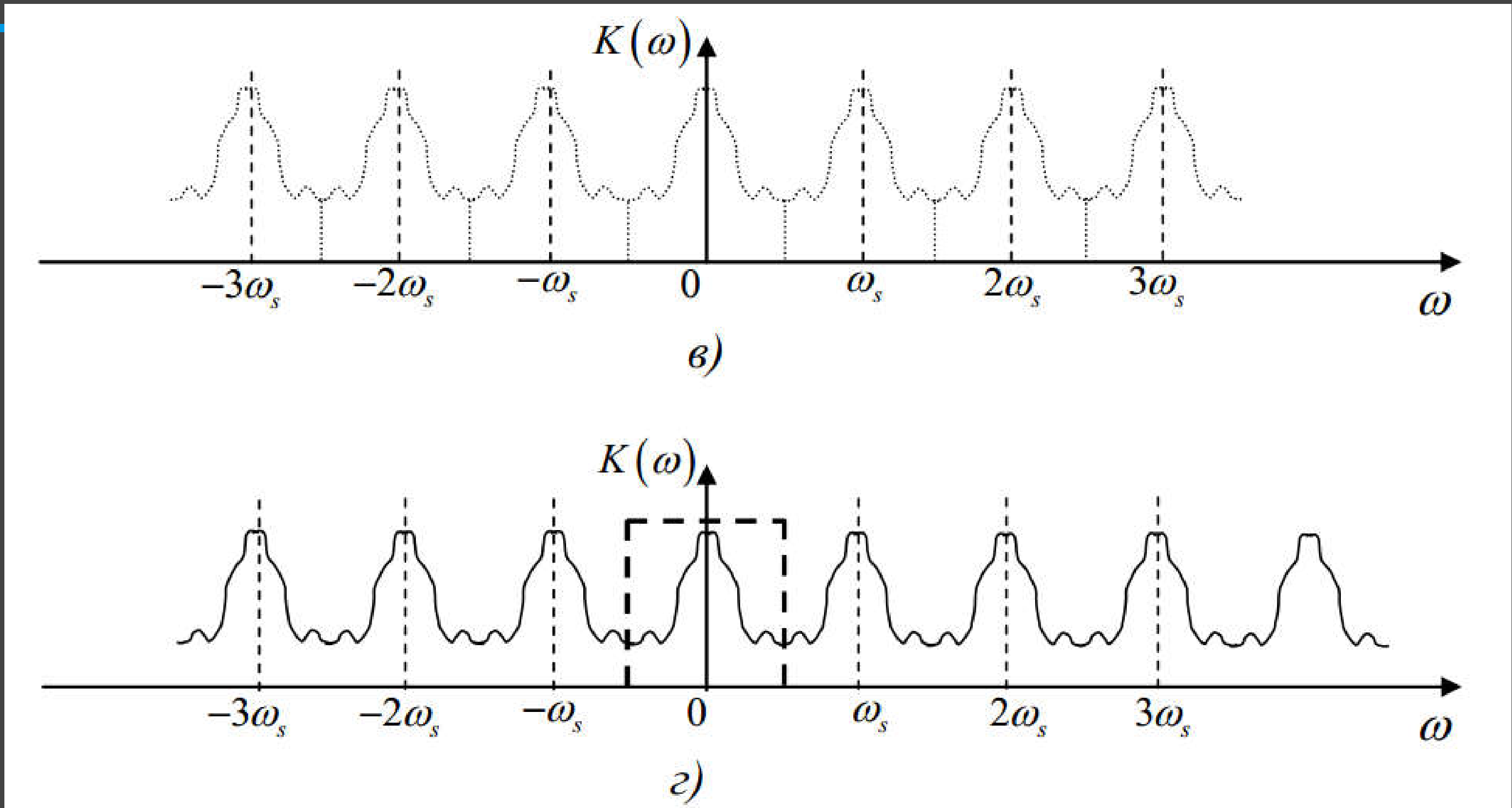


$$\omega_s - \omega_в > \omega_в, \quad \omega_s > 2\omega_в$$

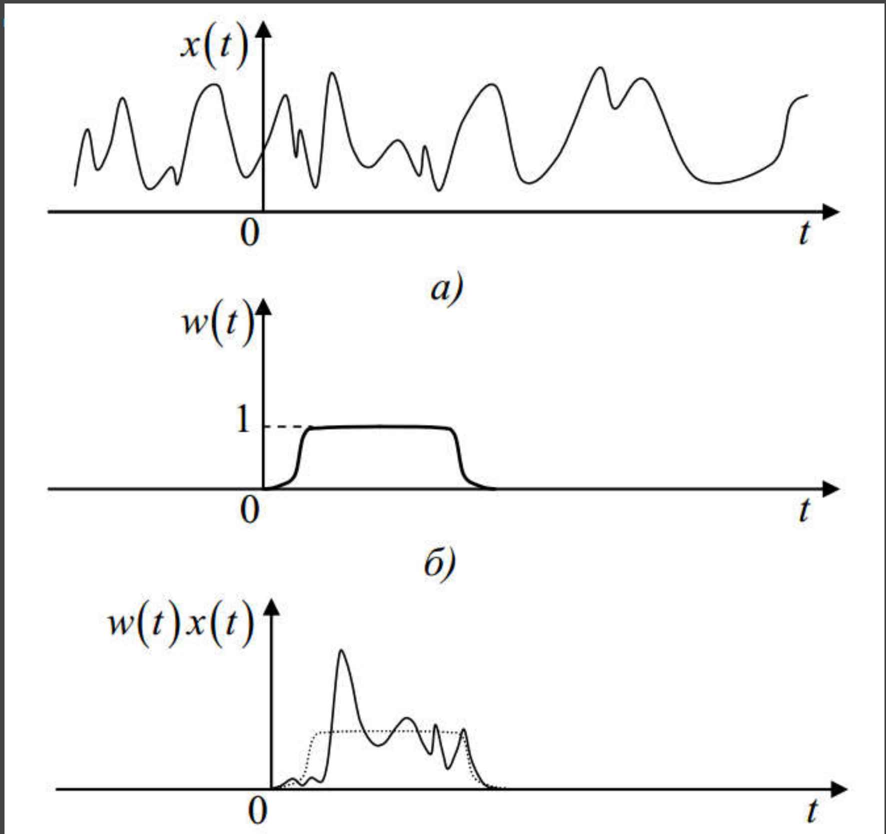
Preliminary filtration – 1



Preliminary filtration – 2

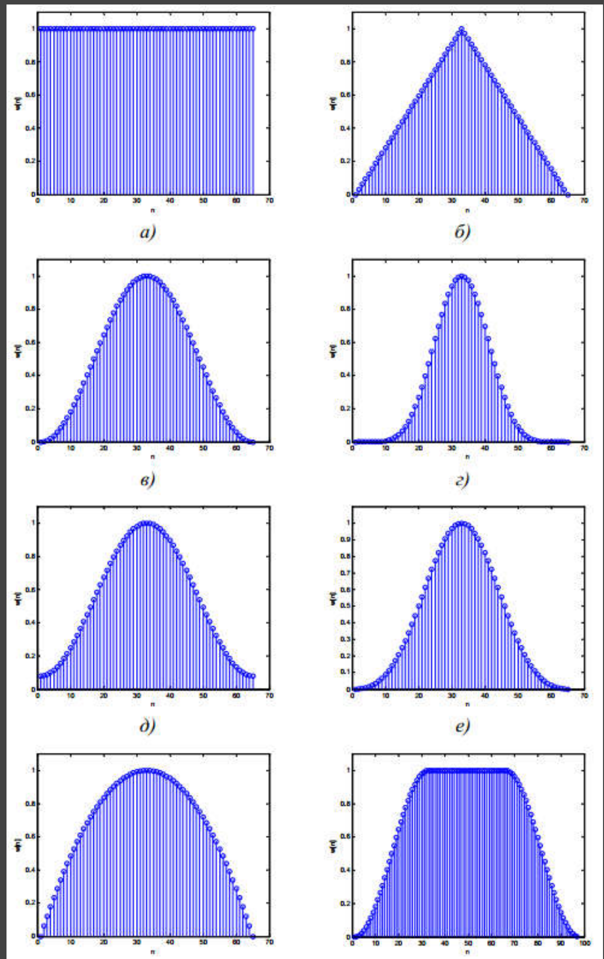


Windowed Fourier Transform – 1

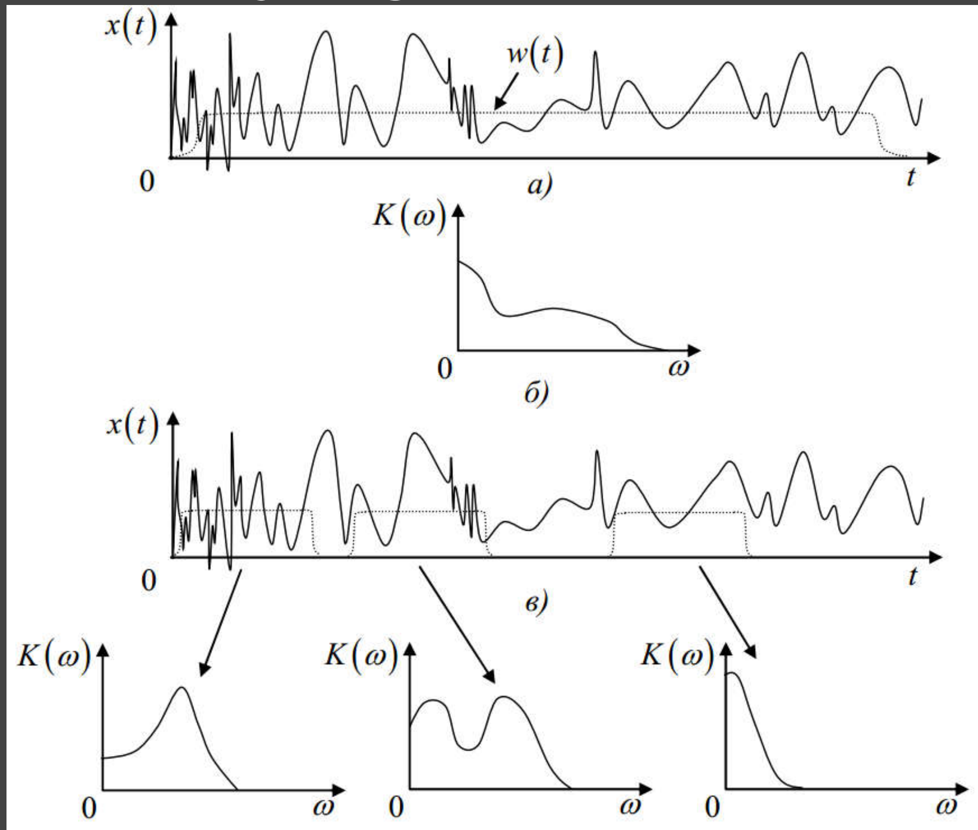


$$F_w(\omega) = \int_{-\infty}^{\infty} x(t)w(t)e^{-j\omega t}dt,$$

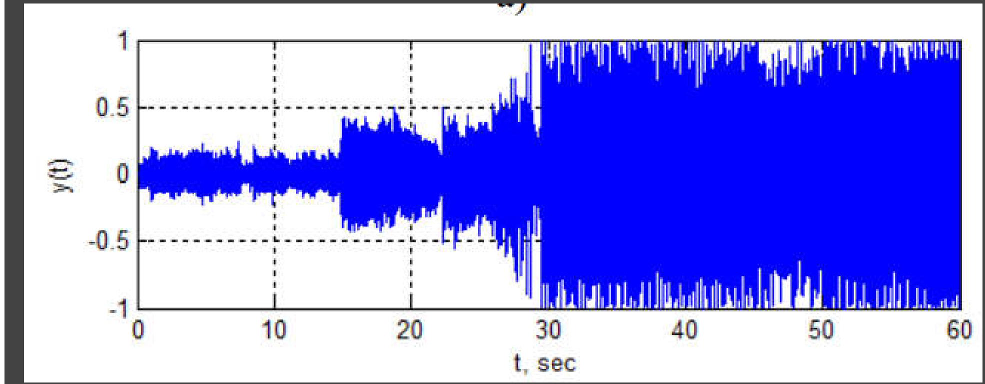
$$c_w[m] = \sum_{n=0}^{N-1} x[n]w[n]e^{-2\pi j\frac{m}{N}n}.$$


а)
б)
в)
г)
д)
е)

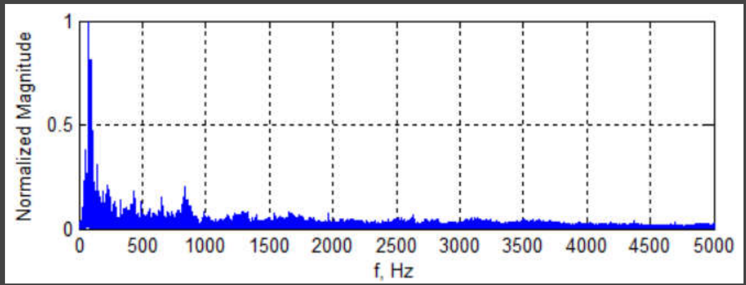
Signals with varying spectrum

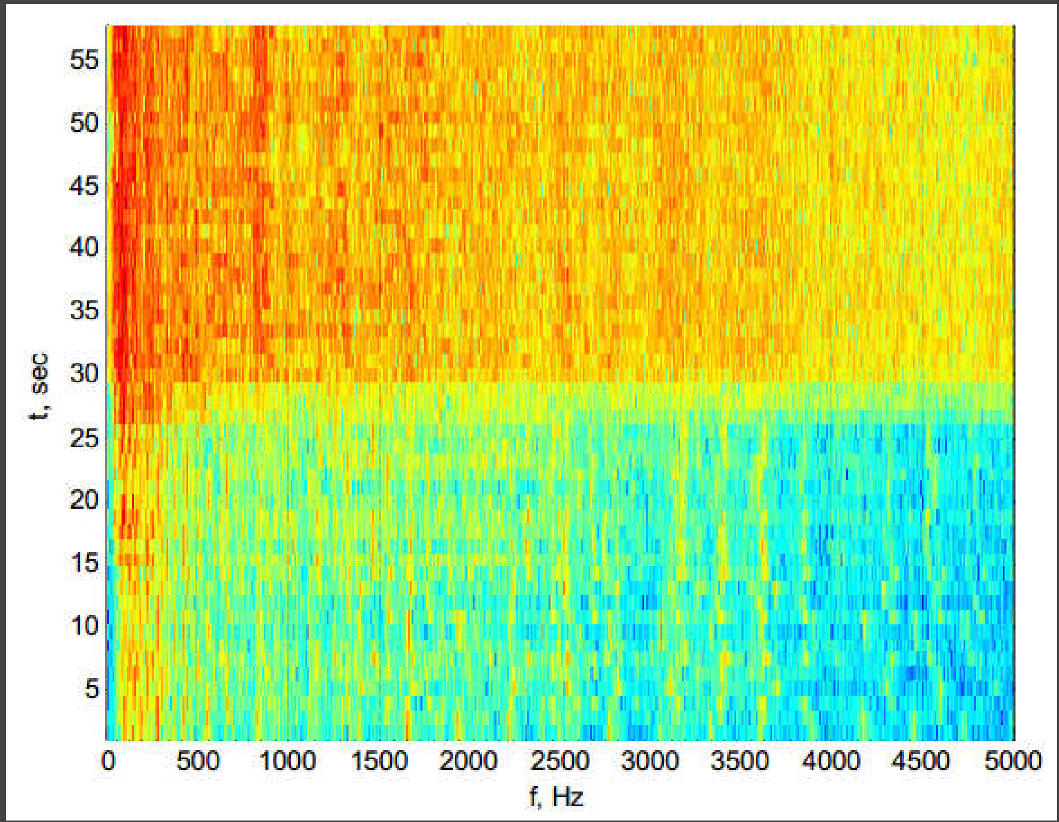


Time-frequency representation – 2



Time-frequency representation – 4





Time-frequency representation – 3

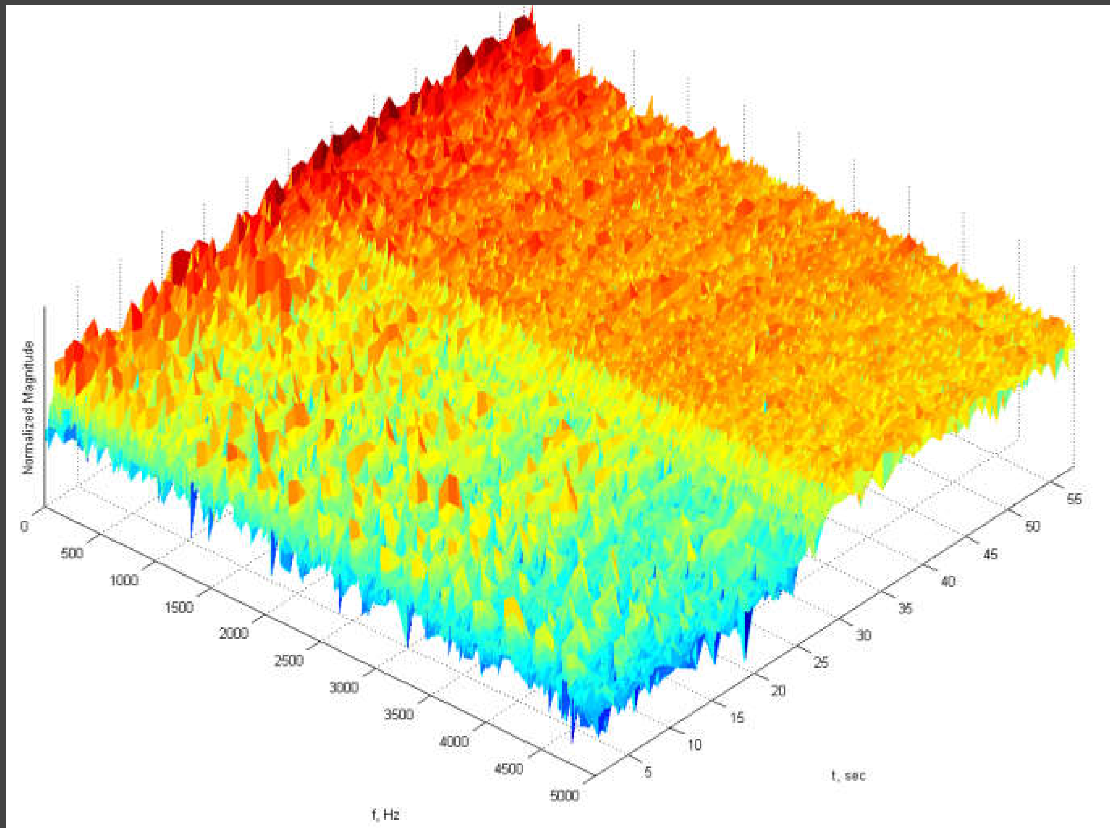
$$F_w(\omega,\tau)=\int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-j\omega t}dt$$

$$c[m,k]=\sum_{n=0}^{N-1} x[n]w[n-k]e^{-2\pi j\frac{m}{N}m}$$

$$S_w(\omega,\tau)=\left|\int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-j\omega t}dt\right|^2,$$

$$C_w[m,k]=\left|\sum_{n=0}^{N-1} x[n]w[n-k]e^{-2\pi j\frac{m}{N}m}\right|^2.$$

Time-frequency representation – 5

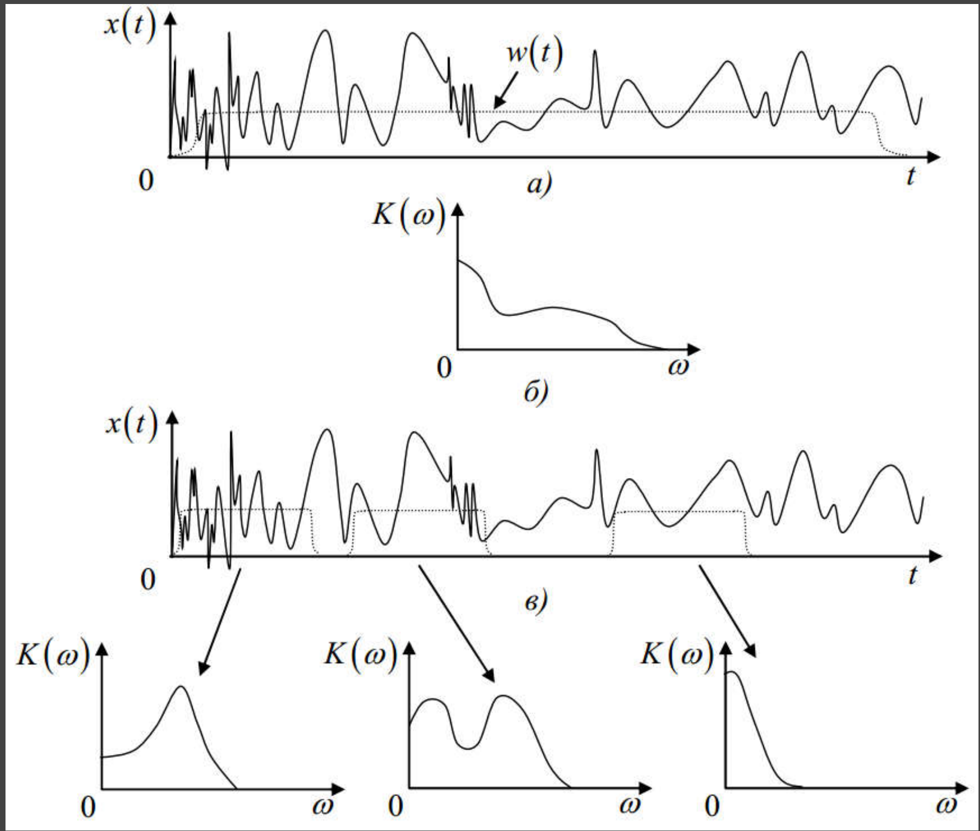


Wavelet transform – 1

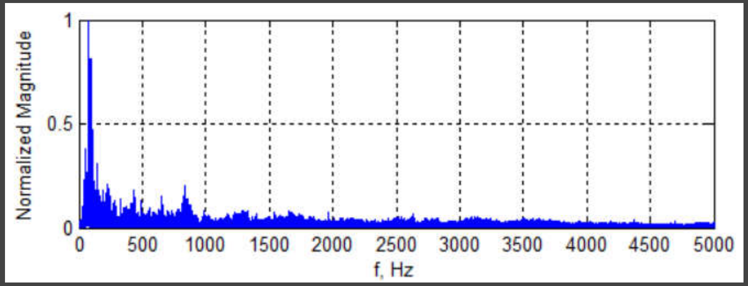
Drawbacks of Fourier transforms:

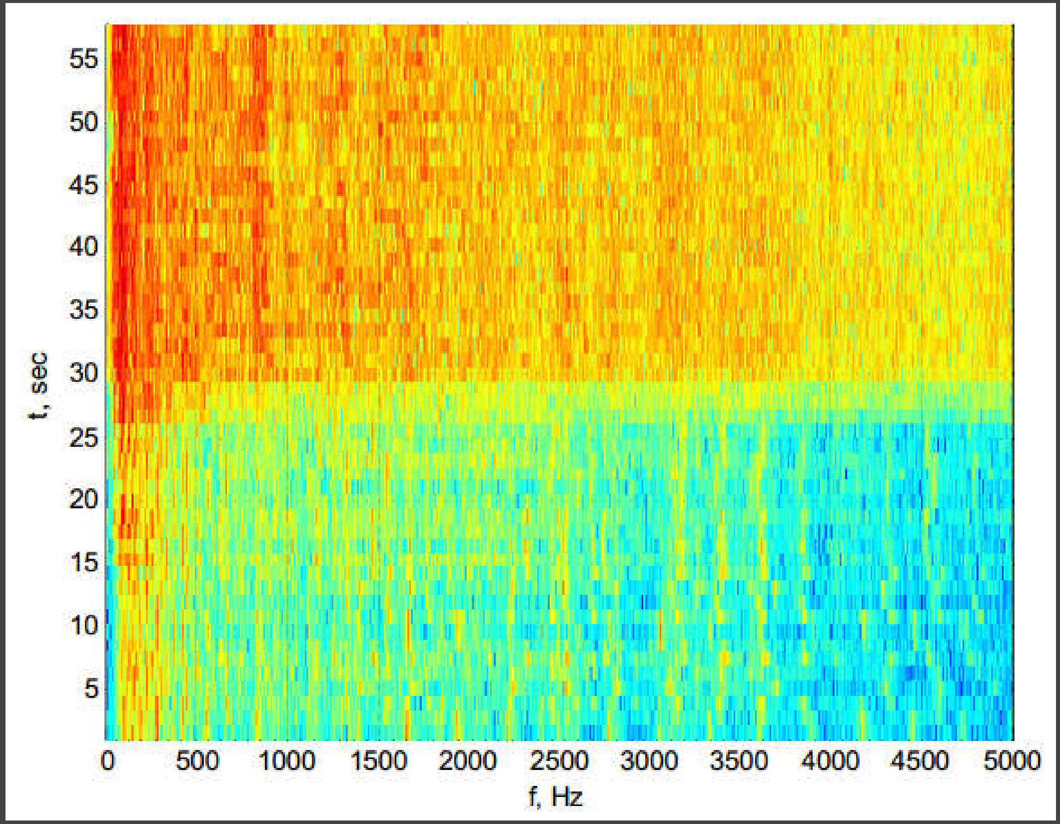
- Unknown intervals of stationarity (window length selection),
- Unknown moments of spectrum change (time shift selection),
- Mixed spectral content (sharp and slow waves).

Signal with time-varying spectra

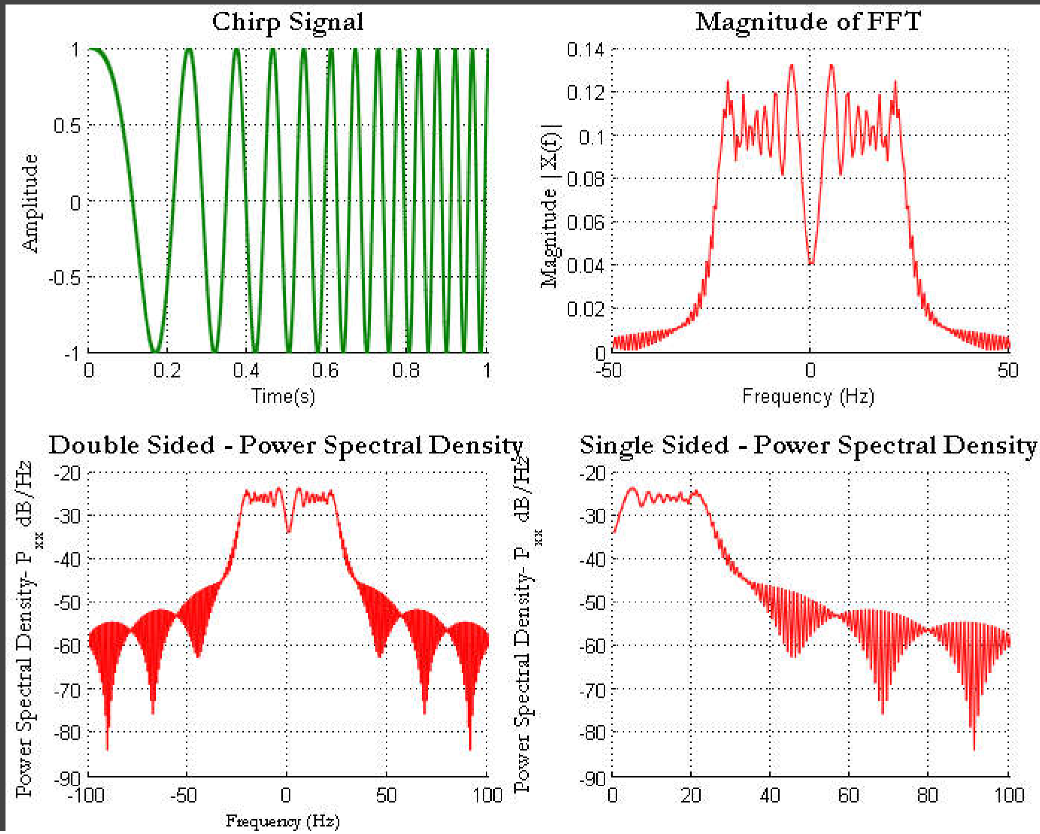


Time-frequency representation

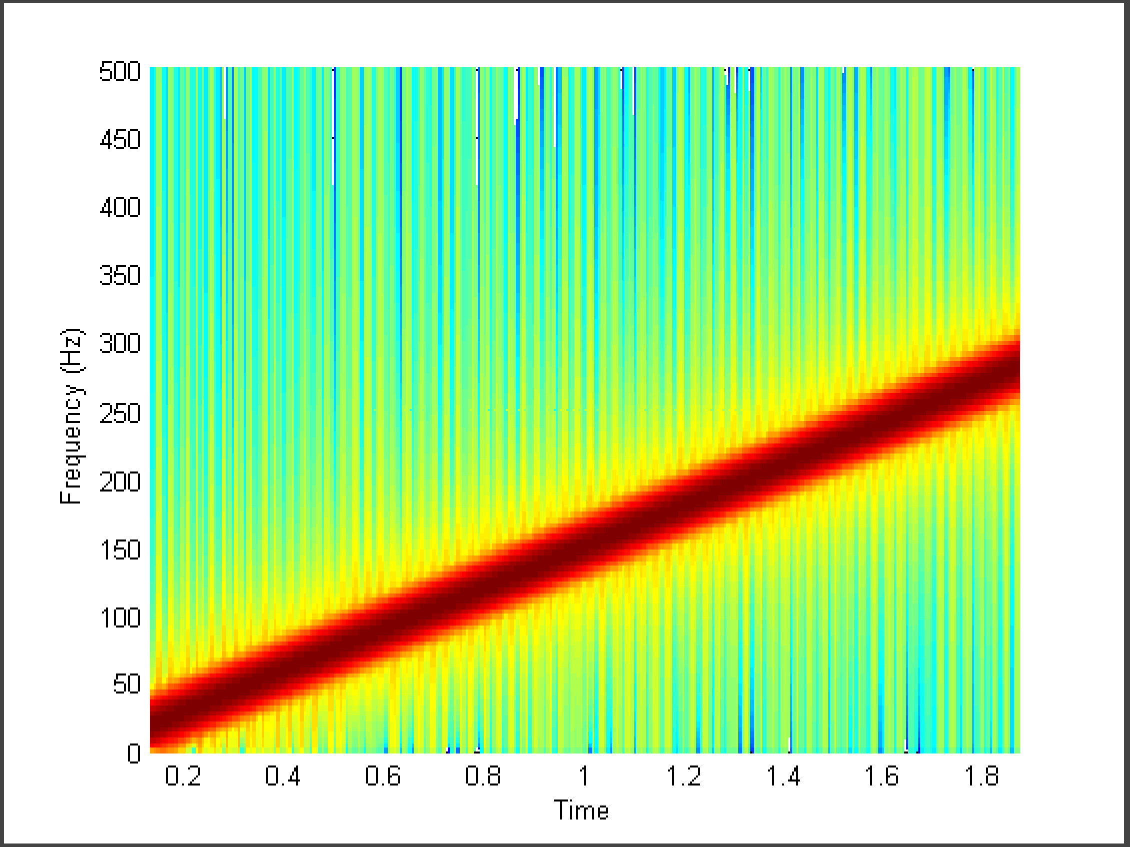




Spectrum of the chirp



Spectrogram of the chirp



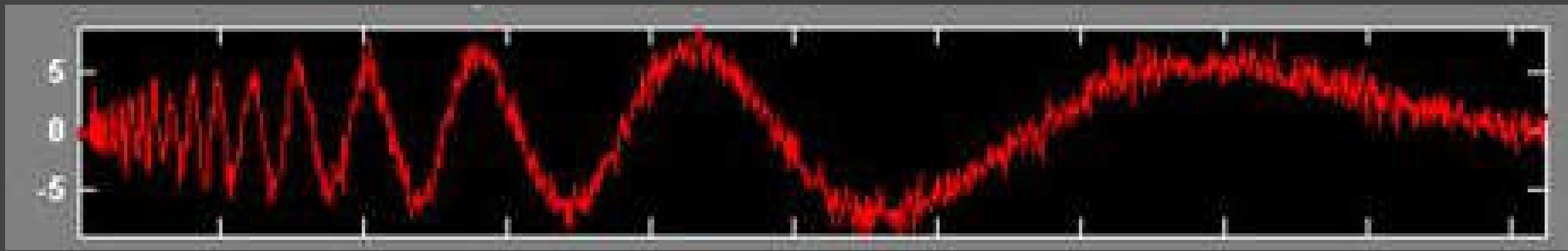
Limitations of Fourier Transform

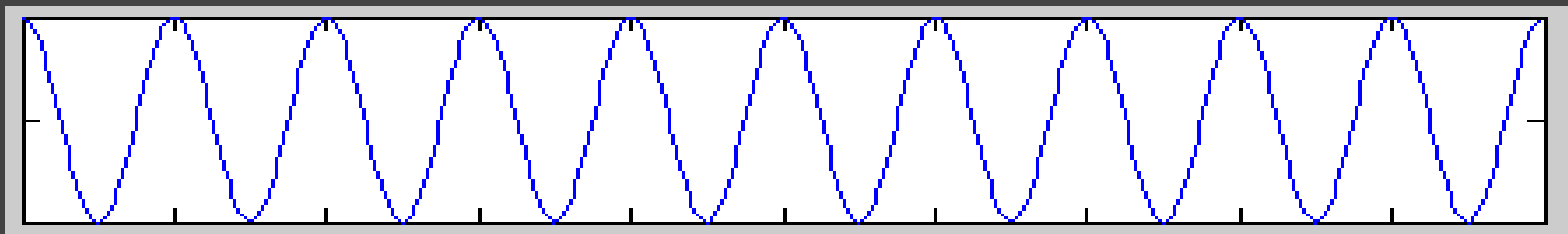
- Unknown intervals of stationarity (window length selection),
- Unknown moments of spectrum change (time shift selection),
- Mixed spectral content (sharp and slow waves).

We loose time information: when in the signal did particular event take place.

We need the possibility to go along the signal and to analyze spectral content in different places separately, and to detect sharp bursts and slow changes.

Changes in spectrum – chirp signal





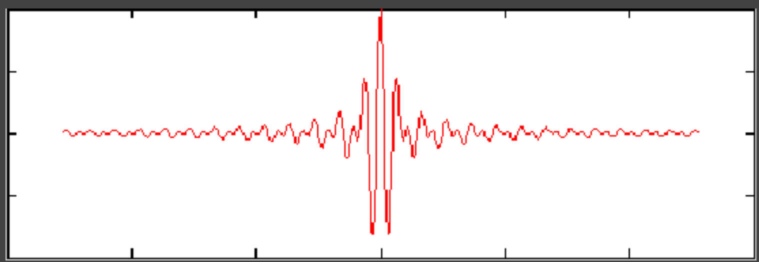
Wavelet Analysis

Wavelet analysis is the decomposition of a signal into a wave-shaped components of constant shape, different durations and time positions.

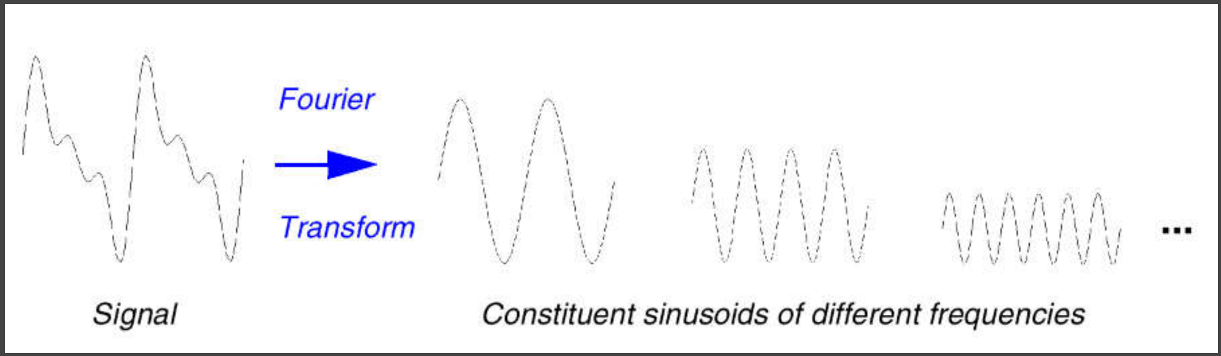
$$x(t) = \sum_n c_n \psi_n(t)$$

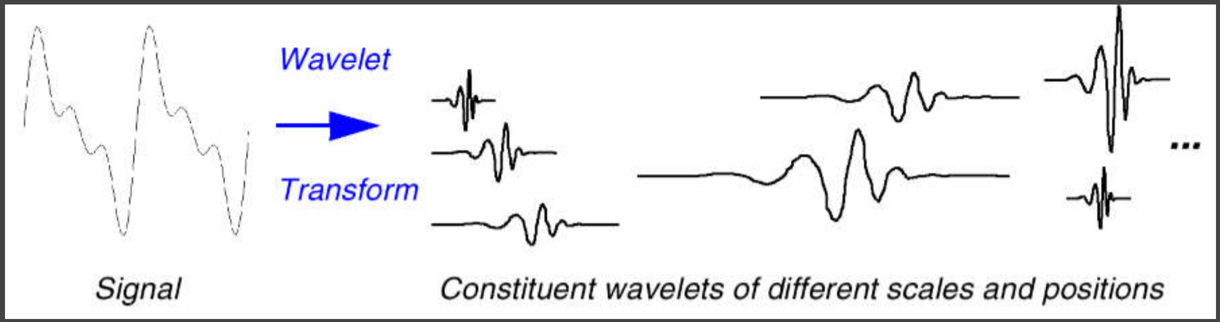
$$C_\psi = \int_0^{+\infty} \frac{|\Psi(\omega)|^2}{\omega} d\omega < +\infty$$

$$\|\psi(t)\| = 1$$



FT vs. WT


Fourier
Transform
Signal
Constituent sinusoids of different frequencies


Wavelet
Transform
Signal
Constituent wavelets of different scales and positions

Continuous Wavelet Transform (CWT) – 1

$$W_\psi(a,\tau) \equiv \langle x(t), \psi_{a,\tau}(t) \rangle =$$

$$= \int_{-\infty}^{\infty} x(t)\psi^*_{a,\tau}(t)dt = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t)\psi^*\left(\frac{t-\tau}{a}\right)dt,$$

$$\psi_{a,\tau}(t) \equiv |a|^{-\frac{1}{2}} \psi\left(\frac{t-\tau}{a}\right)$$

CWT – 2

Calderon-Grossman-Morlet theorem:

$$x(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{R^+/0} \langle x(t), \psi_{a,\tau}(t) \rangle \psi_{a,\tau}(t) \, d\tau \frac{da}{a^2} =$$

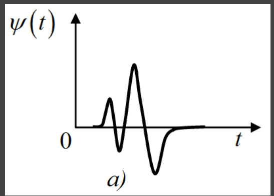
$$= \frac{1}{C_\psi} \int_0^{+\infty} \int_{R^+/0} W_\psi(a,\tau) \frac{1}{\sqrt{a}} \psi\left(\frac{t-\tau}{a}\right) d\tau \frac{da}{a^2}.$$

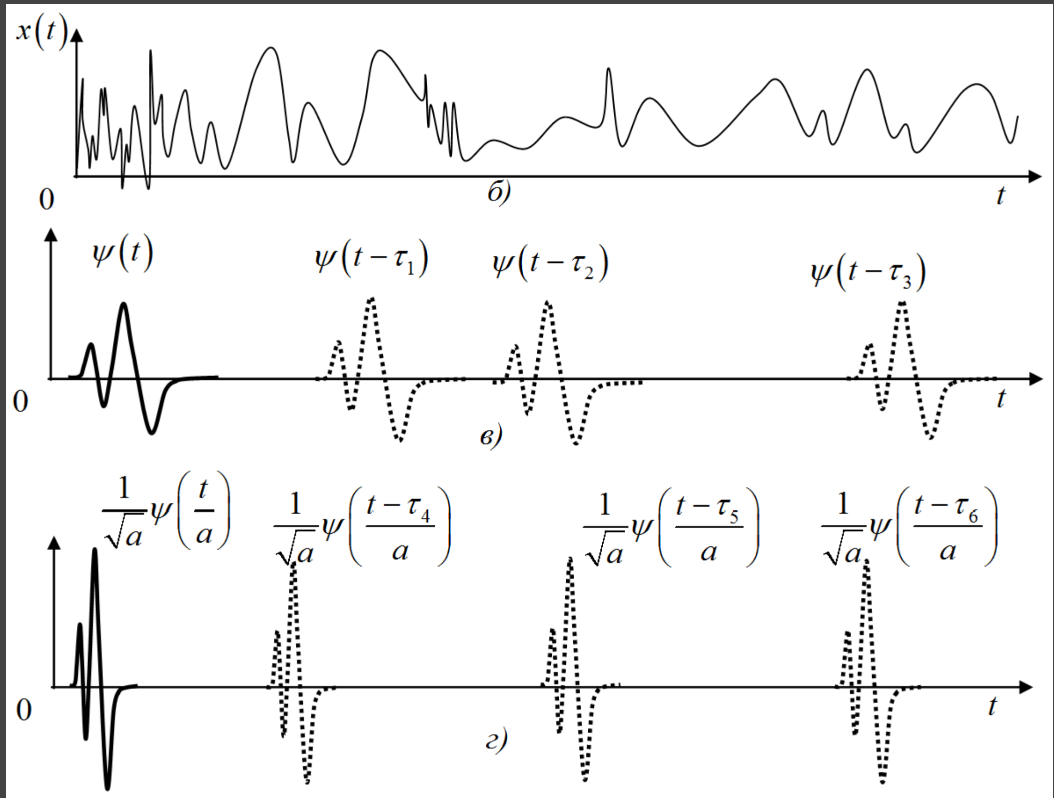
CWT – 3. Scalogram

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{C_\psi} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} |W_\psi(a,\tau)|^2 \, d\tau \, \frac{da}{a^2}$$

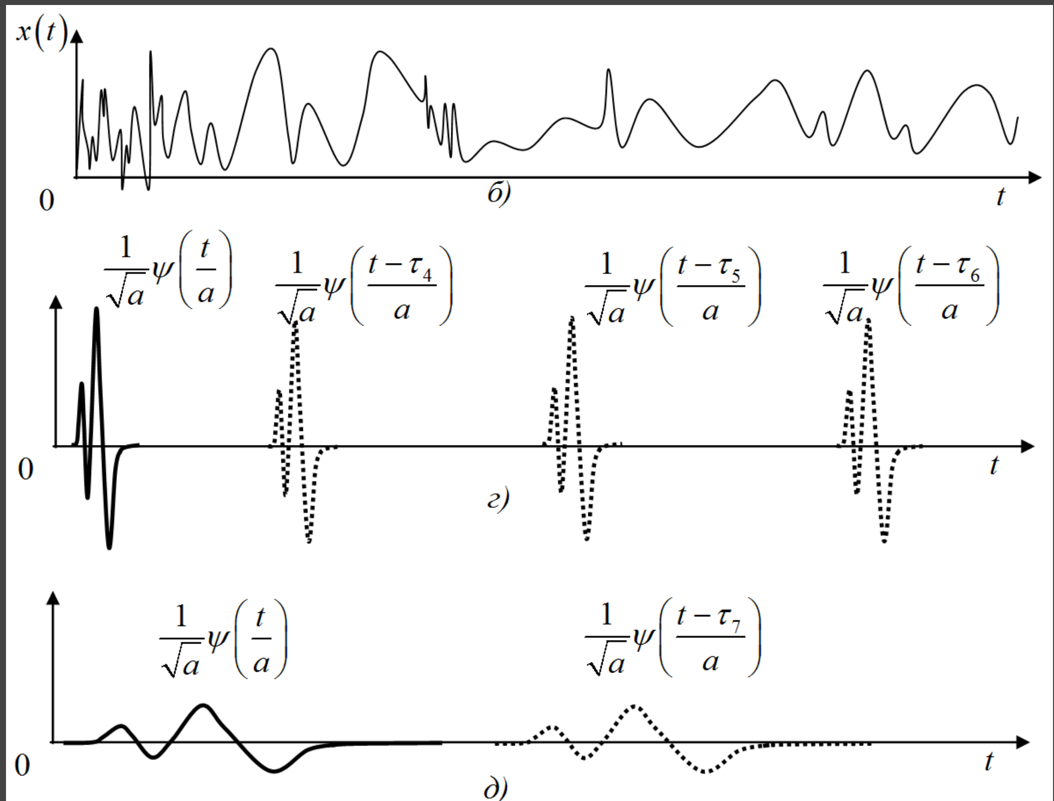
$$E(a,\tau) = |W_\psi(a,\tau)|^2$$

CWT algorithm



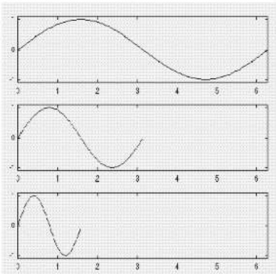


CWT algorithm



Scaling

- Wavelet analysis produces a time-scale view of the signal.
- *Scaling* means stretching or compressing of the signal.
- scale factor (a) for sine waves:

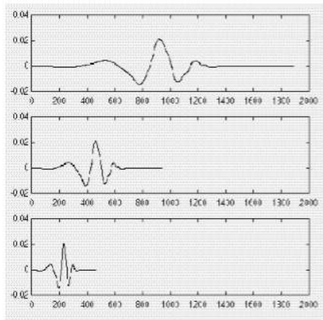


$$f(t) = \sin(t) \;\; ; \;\; a = 1$$

$$f(t) = \sin(2t) \;\; ; \;\; a = \tfrac{1}{2}$$

$$f(t) = \sin(4t) \;\; ; \;\; a = \tfrac{1}{4}$$

- Scale factor works exactly the same with wavelets:



$$f(t) = \Psi(t) \;\; ; \;\; a = 1$$

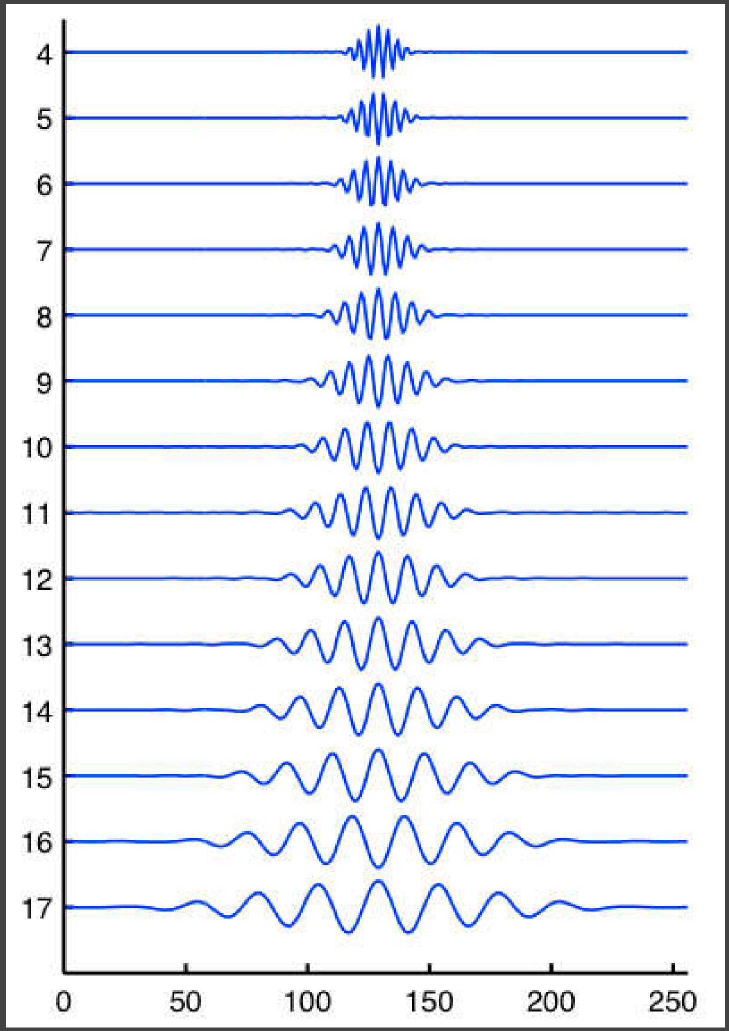
$$f(t) = \Psi(2t) \;\; ; \;\; a = \tfrac{1}{2}$$

$$f(t) = \Psi(4t) \;\; ; \;\; a = \tfrac{1}{4}$$

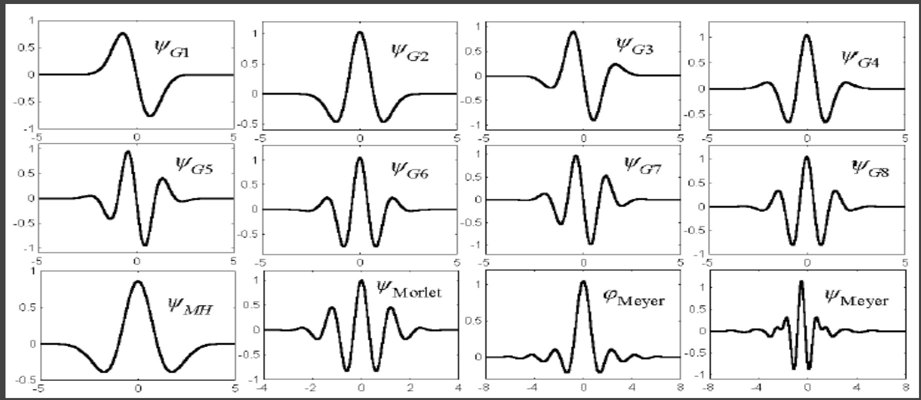
- Scale
  - S>1: dilate the signal
  - S<1: compress the signal
- Low Frequency -> High Scale -> Non-detailed Global View of Signal -> Span Entire Signal
- High Frequency -> Low Scale -> Detailed View Last in Short Time
- Only Limited Interval of Scales is Necessary

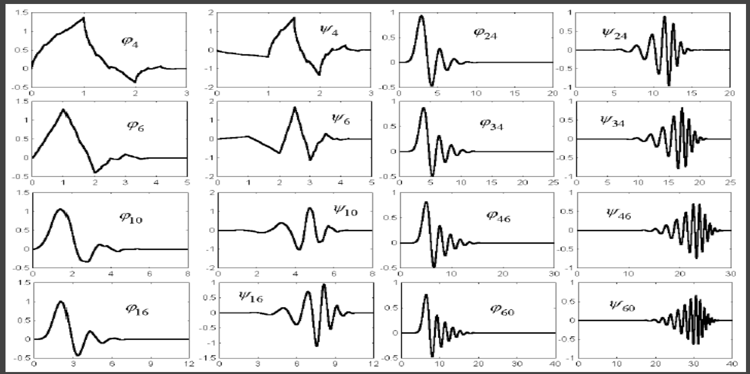
Scaled wavelets

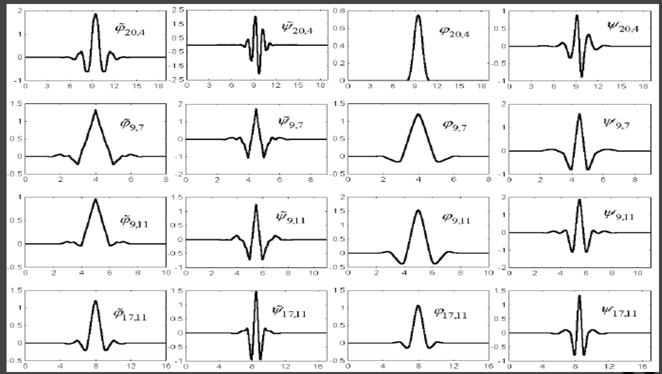
Compactly supported components



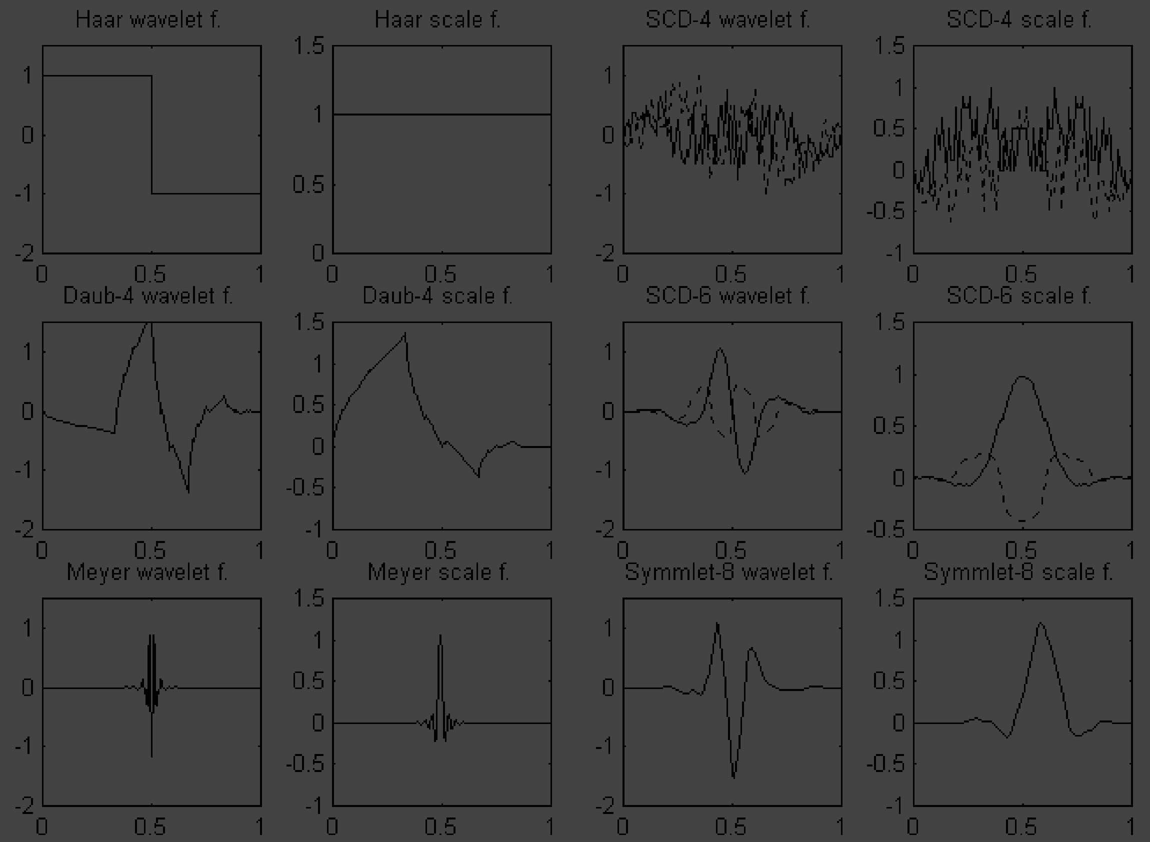
Mother Functions



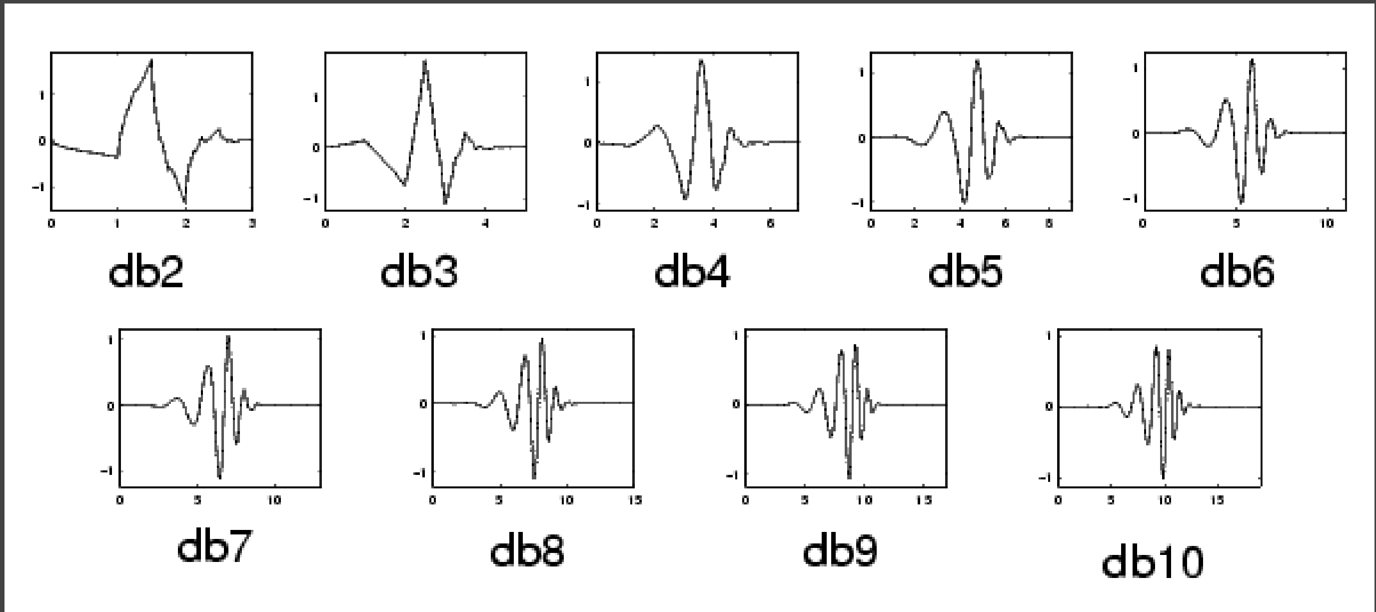




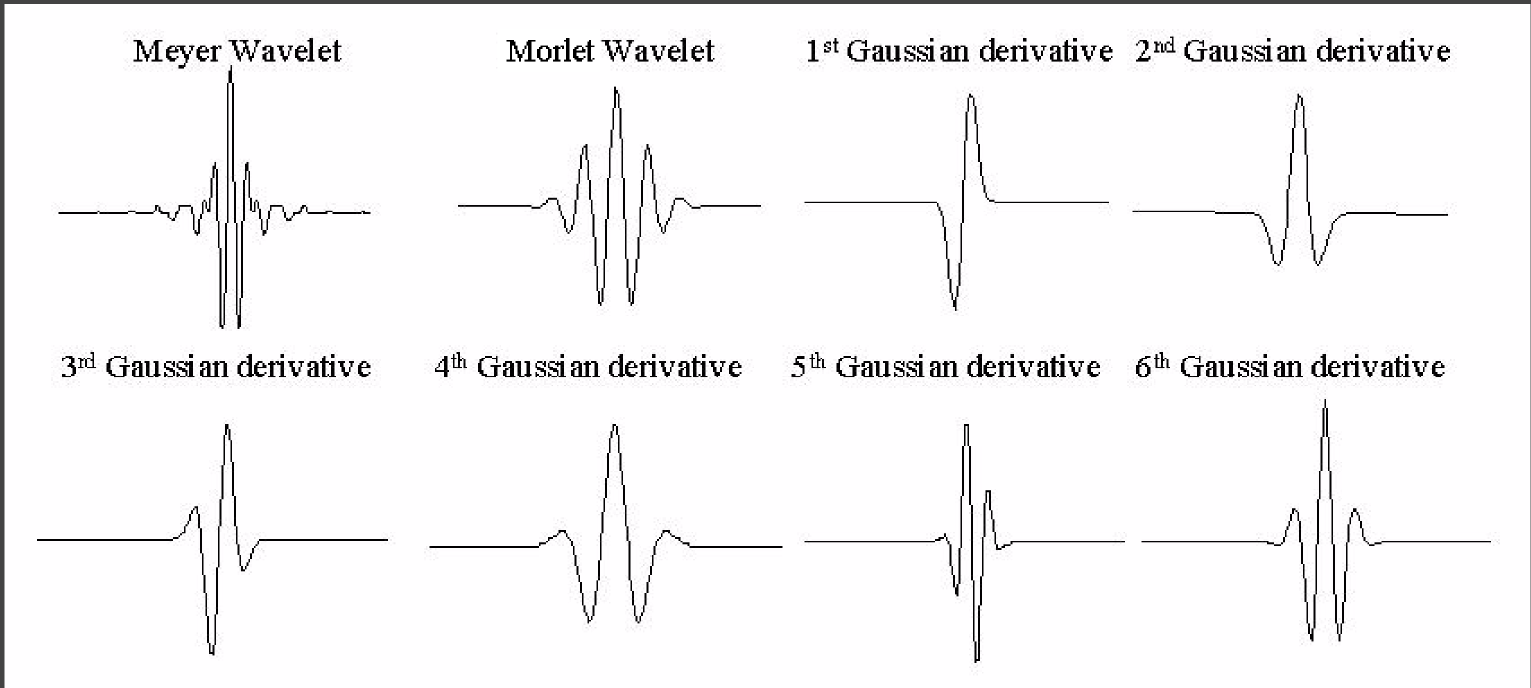
Wavelet mother functions



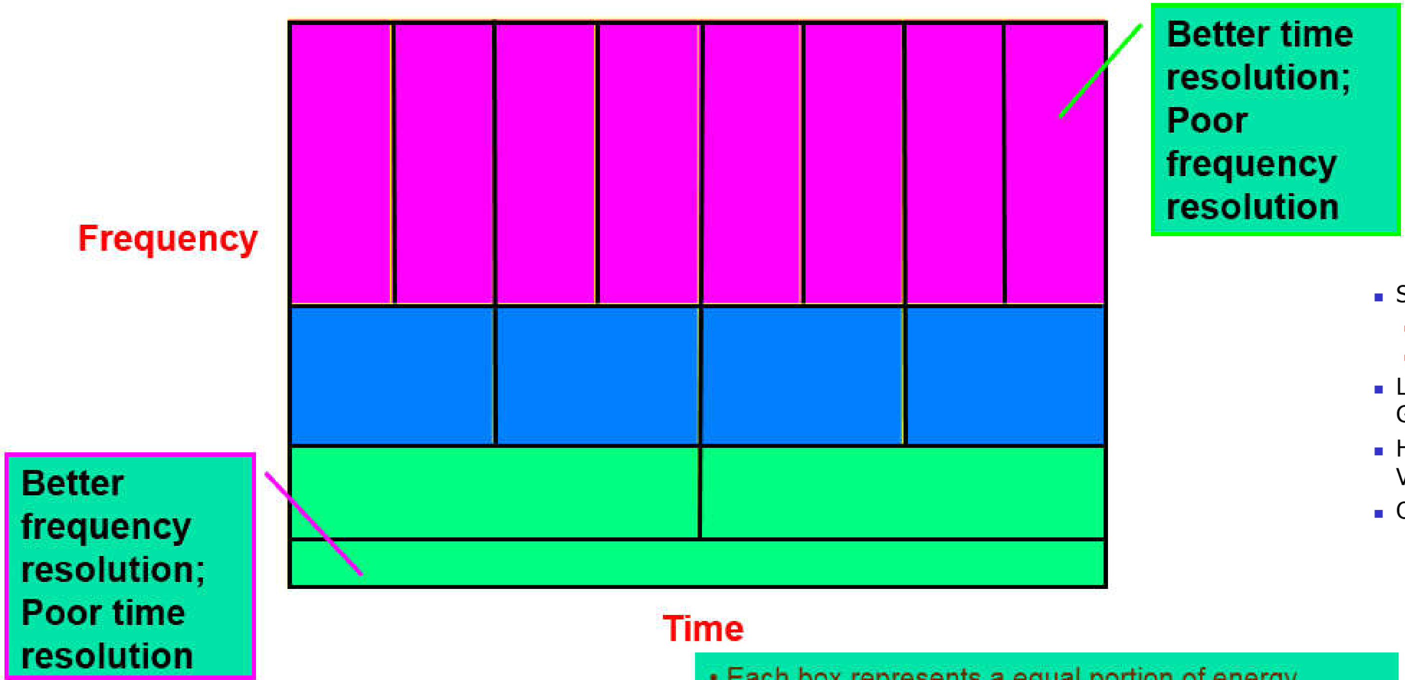
Daubechies wavelet family



Gaussian Wavelets



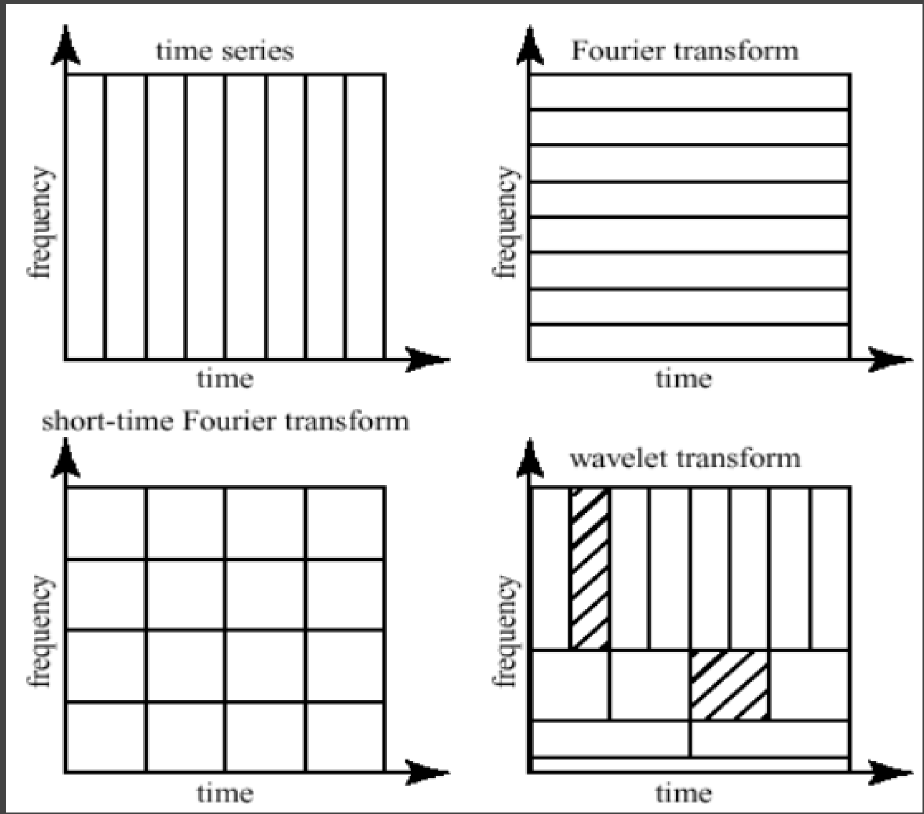
Time-scale resolution



- Each box represents a equal portion of energy
- *Resolution in STFT is selected once for entire analysis*

- Scale
  - S>1: dilate the signal
  - S<1: compress the signal
- Low Frequency -> High Scale -> Non-detailed Global View of Signal -> Span Entire Signal
- High Frequency -> Low Scale -> Detailed View Last in Short Time
- Only Limited Interval of Scales is Necessary

Time-scale (frequency) resolution



CWT discretization

- It is Necessary to Sample the Time-Frequency (scale) Plane.
- At High Scale s (Lower Frequency f), the Sampling Rate N can be Decreased.
- The Scale Parameter s is Normally Discretized on a Logarithmic Grid.
- The most Common Value is 2.
- The Discretized CWT is not a True Discrete Transform

- Discrete Wavelet Transform (DWT)
  - Provides sufficient information both for analysis and synthesis
  - Reduce the computation time sufficiently
  - Easier to implement
  - Analyze the signal at different frequency bands with different resolutions
  - Decompose the signal into a coarse approximation and detail information

Discrete Wavelet Transform

- Suppose that $x[n]$ is a square-summable sequence, that is $x[n] \in \ell_2(\mathbb{Z})$
- Orthonormal expansion of $x[n]$ is of the form

$$x[n] = \sum_{k\in\mathbb{Z}} \langle \varphi_k[l], x[l] \rangle \varphi_k[n] = \sum_{k\in\mathbb{Z}} X[k]\varphi_k[n] \Longrightarrow \|x\|^2 = \|X\|^2$$

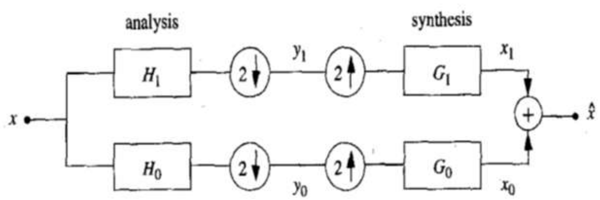
- Where $X[k] = \langle \varphi_k[l], x[l] \rangle = \sum_l \varphi_k^*[n] x[l]$

  is the transform of $x[n]$
- The basis functions φ_k satisfy the orthonormality constraint

$$\langle \varphi_k[n], \varphi_l[n] \rangle = \delta[k-l]$$

Two-channel filter banks

- Filter bank is the building block of discrete-time wavelet transform
- For 1-D signals, two-channel filter bank is depicted below



- For perfect reconstruction filter banks we have

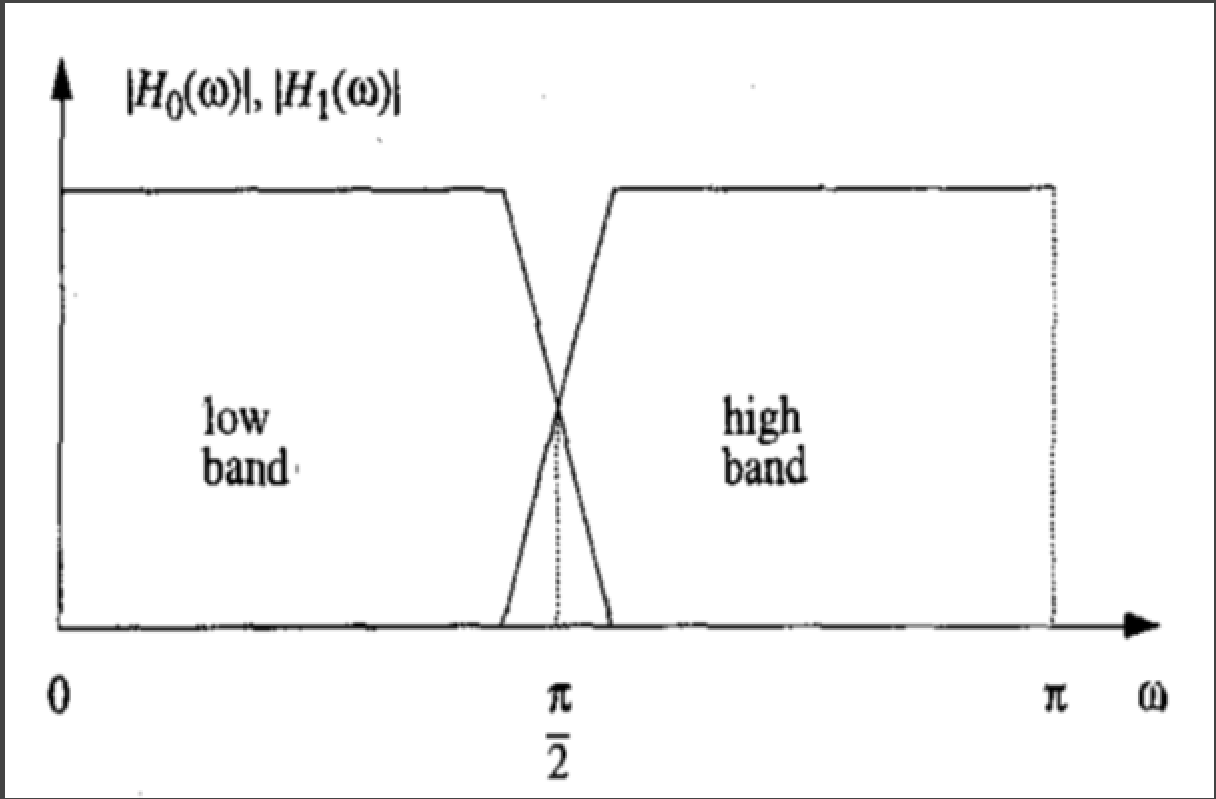
$$\hat{x} = x$$

- In order to achieve perfect reconstruction the filters should satisfy

$$\begin{cases} g_0[n] = -h_0[-n] \\ g_1[n] = h_1[-n] \end{cases}$$

- Thus if one filter is lowpass, the other one will be highpass

FR of filter pair



Orthogonal Filters

- To have orthogonal wavelets, the filter bank should be orthogonal
- The orthogonal condition for 1-D two-channel filter banks is

$$g_1[n] = (-1)^n g_0[-n+1]$$

- Given one of the filters of the orthogonal filter bank, we can obtain the rest of the filters

Haar filter bank

- The simplest orthogonal filter bank is Haar
- The lowpass filter is

$$h_0[n] = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0, -1 \\ 0, & \text{otherwise} \end{cases}$$

- And the highpass filter

$$h_1[n] = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0 \\ -\frac{1}{\sqrt{2}}, & n = -1 \\ 0, & \text{otherwise} \end{cases}$$

- The lowpass output is

$$y_0[k] = h_0[n] * x[n]\big|_{n=2k} = \sum_{l \in \mathbb{Z}} h_0[l]x[2k-l] = \frac{1}{\sqrt{2}}x[2k] + \frac{1}{\sqrt{2}}x[2k+1]$$

- And the highpass output is

$$y_1[k] = h_1[n] * x[n]\big|_{n=2k} = \sum_{l \in \mathbb{Z}} h_1[l]x[2k-l] = \frac{1}{\sqrt{2}}x[2k] - \frac{1}{\sqrt{2}}x[2k+1]$$

Time-reversal filters

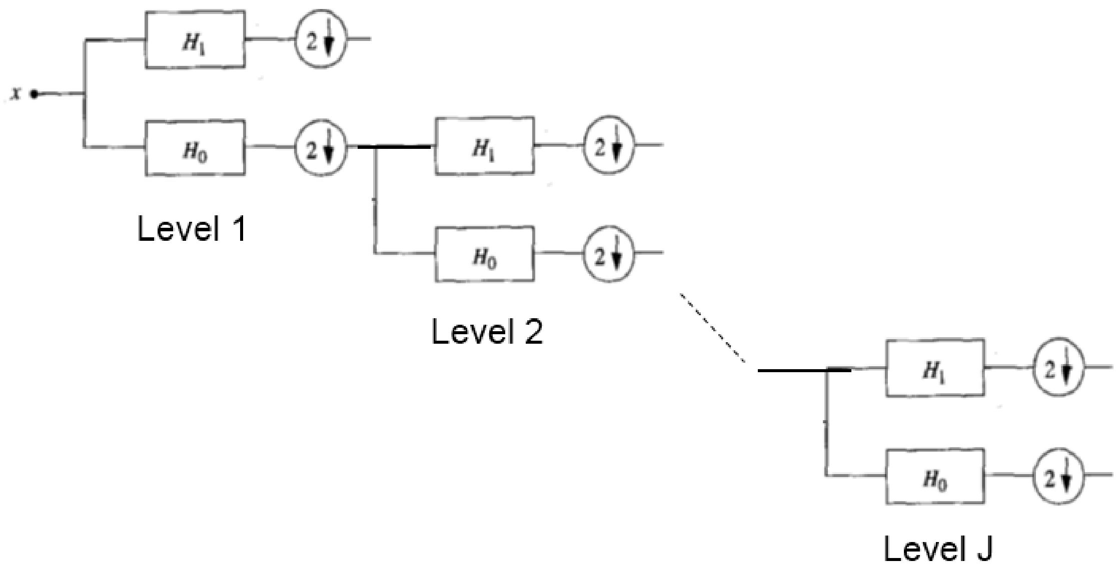
- Since $y_0[k] = X[2k]$ and $y_1[k] = X[2k+1]$, the filter bank implements Haar expansion
- Note that the analysis filters are time-reversed versions of the basis functions

$$h_0[n] = \varphi_0[-n] \qquad h_1[n] = \varphi_1[-n]$$

since convolution is an inner product followed by time-reversal

DWT – 1

- We can construct discrete WT via iterated (octave-band) filter banks
- The analysis section is illustrated below



DWT – 2

- And the synthesis section is illustrated here
- If $h_i[n]$ is an orthogonal filter and $g_i[n] = h_i[-n]$, then we have an orthogonal wavelet transform